1

PRE-STUDY FOR OPERATIONS RESEARCH

1.1 INTRODUCTION

The main purpose of this chapter is to provide with basic concepts of preliminary mathematics required for understanding this new subject of Operations Research. Due to wide scope of O R in various disciplines, some useful topics of elementary mathematics are discussed in this chapter. It covers some important concepts on : ***Simultaneous linear equations, Differentiation and integration, Generating functions, Finite differences,*** etc.

The mathematical methods are presented in this chapter with a minimum mathematical complexity. However, this approach does not reduce the potential value of these methods.

I—Vectors and Linear Algebra

1.2 VECTOR INEQUALITIES

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. Then **a** is said to be ***greater than or equal to*** **b**, if and only if, $a_1 \geq b_1, a_2 \geq b_2, \ldots, a_n \geq b_n$. This is denoted by $\mathbf{a} \geq \mathbf{b}$.

Similarly, if $a_1 \leq b_1, a_2 \leq b_2, \ldots, a_n \leq b_n$, then **a** is said to be ***less than or equal to*** **b**, denoted by $\mathbf{a} \leq \mathbf{b}$.

Two vectors **a** and **b** are said to be ***order comparable***, if and only if, they have equal number of elements.

Example. Let

$\mathbf{a} = (1, 2, 3), \mathbf{b} = (0, 1, 2)$. Clearly, $\mathbf{a} \geq \mathbf{b}$.

1.3 LINEAR COMBINATION OF VECTORS

Suppose we are given two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. The operations of addition and scalar multiplication can be applied to obtain the expressions like $\mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 - \mathbf{a}_2, 2\mathbf{a}_1 + 3\mathbf{a}_2, 3\mathbf{a}_1 - 5\mathbf{a}_2$, etc. Such vectors are called ***linear combination of vectors*** $\mathbf{a}_1$ and $\mathbf{a}_2$. In general, the vector $\mathbf{b} = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. Figure 1·1 explains this point.

Thus, if $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ is a set of k vectors in n-space and $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is a set of k scalars, then the vector

$$\mathbf{b} = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \ldots + \lambda_k\mathbf{a}_k$$

is a linear combination of the given set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$.

This concept of linear combination provides us with a definition of the line segment between two vectors, or points.

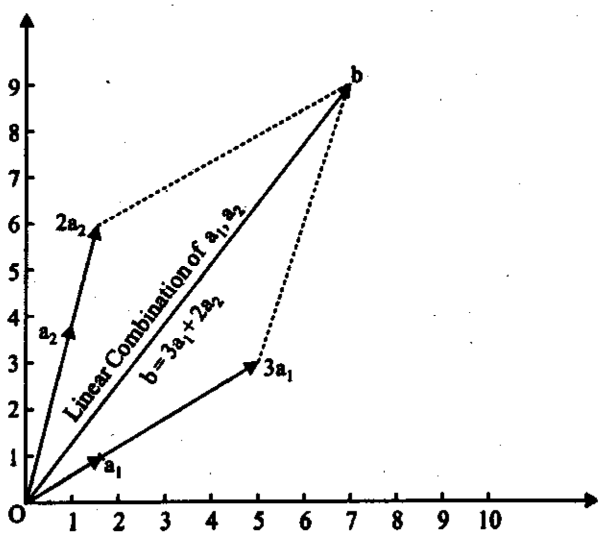


Fig. 1.1
Vector b as Linear Combination of $\mathbf{a}_1, \mathbf{a}_2$

Line segment between $\mathbf{a}_1$ and $\mathbf{a}_2$:

Definition. The line segment between two points (vectors) $\mathbf{a}_1$ and $\mathbf{a}_2$ is the set of points

$$\mathbf{b} = \lambda \mathbf{a}_1 + (1 - \lambda)\, \mathbf{a}_2 \text{, for all } \lambda,\ 0 \le \lambda \le 1 .$$

This definition of the line segment can be extended to any number of points. In its more general form, this linear combination is called a ***convex combination*** and represents a segment of a plane. The plane segment is called the ***convex-hull*** of the points.

Convex–Hull of points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ in E^n.

Definition. The convex-hull of p points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ in E^n is the *set of points* $\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_p \mathbf{a}_p$, for all *non-negative* $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that $\lambda_1 + \lambda_2 + \ldots + \lambda_p = 1$.

Example 1. *Consider three points* $a_1 = (6, 6)$, $\mathbf{a}_2 = (9, 12)$, $\mathbf{a}_3 = (3, 9)$.

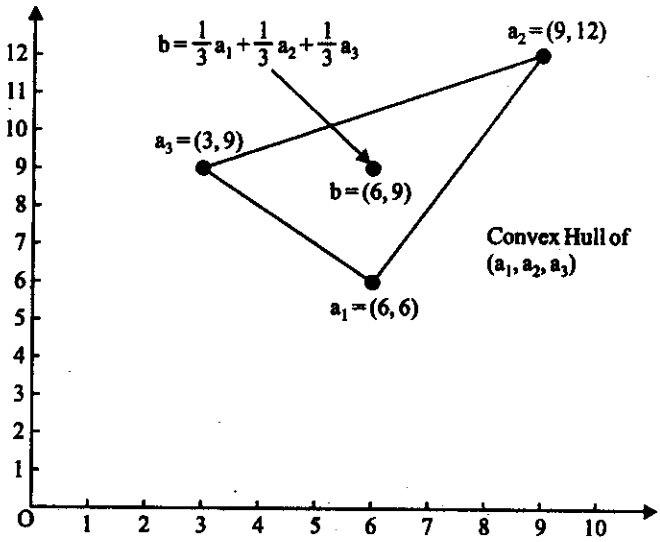


Fig. 1.2
Convex hull of three points in E^2

The convex hull of these three points is shown in the side Fig. 1·2. If $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{1}{3}, \lambda_3 = \frac{1}{3}$ (so that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and all $\lambda \ge 0$), then $\mathbf{b} = \frac{1}{3}(6, 6) + \frac{1}{3}(9, 12) + \frac{1}{3}(3, 9) = (6, 9)$.

1.4 LINEAR INDEPENDENCE, SPANNING SET AND BASIS

We now define some useful properties of set of vectors. If, in E^n, we have a set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$, we say that they are ***linearly independent*** if no one of these can be expressed as a linear combinations of the remaining ones.

Linear Independence. A set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ is linearly independent if the equation

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_k \mathbf{a}_k = 0,$$

is satisfied only if $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$.

A set of vectors is *linearly dependent* if it is not linearly independent.

Example 1. *The vectors* $\mathbf{a}_1 = (1, 2)$ and $\mathbf{a}_2 = (2, 4)$ *are linearly dependent since there exist* $\lambda_1 = 2$ and $\lambda_2 = -1$ *for which* $\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 = \mathbf{0}$.

Example 2. *The vectors* $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ *and* $e_3 = (0, 0, 1)$ *are linearly independent, since*

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 = \mathbf{0}$$
$$\Rightarrow \lambda_1 (1, 0, 0) + \lambda_2 (0, 1, 0) + \lambda_3 (0, 0, 1) = (0, 0, 0)$$
$$\Rightarrow (\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$$
$$\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0.$$

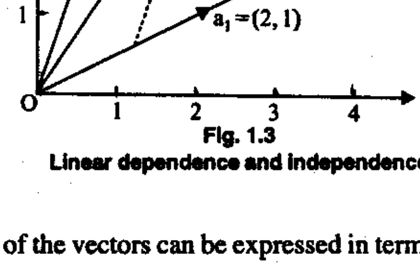


Fig. 1.3
Linear dependence and independence

Example 3. *Consider the set of four vectors* $\mathbf{a}_1 = (2, 1)$, $\mathbf{a}_2 = (1, 3)$, $\mathbf{a}_3 = (2, 3)$, $\mathbf{a}_4 = (4, 2)$. *It can be geometrically seen below that—*

(*i*) The set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent, since neither of the vectors can be expressed in terms of the others.

(*ii*) The set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is not linearly independent, since $\mathbf{a}_3$ can be expressed as the linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

(iii) The set $\{\mathbf{a}_1, \mathbf{a}_4\}$ is also linearly dependent, since $2\mathbf{a}_1 = \mathbf{a}_4$.

(iv) The set of single vector $\{\mathbf{a}_1\}$ is also linearly independent.

Remark. If a set of vectors is linearly independent, then any subset of it is also linearly independent. If a set of vectors is linearly dependent, then a superset of it is also linearly dependent.

Spanning set. The set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ in E^n is a *spanning set* in E^n if every vector in E^n can be expressed as a linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$.

From the Fig. 1.3, it can be seen that no less than two vectors are required to form the spanning set in E^2. So, for example, the set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a spanning set and so is $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. However, the set $\{\mathbf{a}_1\}$ is not a spanning set and neither is $\{\mathbf{a}_1, \mathbf{a}_4\}$ since many vectors in the space cannot be expressed as linear combination of either of these sets.

Basis set. A set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ in E^n is a ***basis set*** if—

(i) *it is a linearly independent set, and* (ii) *it is a spanning set of* E^n.

If it is a basis then $k = n$.

Standard basis. The set of unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is called the standard basis for E^n (since the set of vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is a linearly independent set, and it is also a spanning set of E^n).

Remark. It will be seen later that the standard basis is the same as the identity matrix which will be the basis for the development of simplex method of linear programming. This may be renamed as basis matrix also.

II-Linear Simultaneous Equations

1.5 SIMULTANEOUS EQUATIONS : NATURE OF SOLUTIONS IN DIFFERENT CASES

Let us consider the problem of finding values of variables x_1 and x_2 satisfying simultaneously the linear equations :

$$x_1 + 2x_2 = 8 \text{ and } 3x_1 + x_2 = 9 \qquad \ldots(1\cdot1)$$

Definitions. A pair of values for x_1 and x_2 satisfying (1·1) is called a *solution* to the equations. A unique solution exists when only one pair (x_1, x_2) will satisfy the equations and there is *no solution* if no pair of values satisfies the equations.

First we see the problem (1·1) graphically. We plot both the equations in the graph as given in Fig. 1.4.

The problem demands for those values of x_1 and x_2 that are common to both the equations (lines) : $x_1 + 2x_2 = 8$ and $3x_1 + x_2 = 9$. In this case, the answer is a single point $A = (2, 3)$. The solution values are $x_1 = 2$ and $x_2 = 3$.

Next consider another set of equations :

$$x_1 + 2x_2 = 8,\ 3x_1 + x_2 = 9 \text{ and } x_1 + x_2 = 4 \qquad \ldots(1\cdot2)$$

As seen in Fig. 1·4, no pair (x_1, x_2) exists that satisfies all the three equations simultaneously. So there is no solution to (1·2). Such simultaneous equations are called ***inconsistent***.

Now changing the last equation of (1·2) by $x_1 + x_2 = 5$, the system becomes :

$$x_1 + 2x_2 = 8,\ 3x_1 + x_2 = 9,\ x_1 + x_2 = 5 \qquad \ldots(1\cdot3)$$

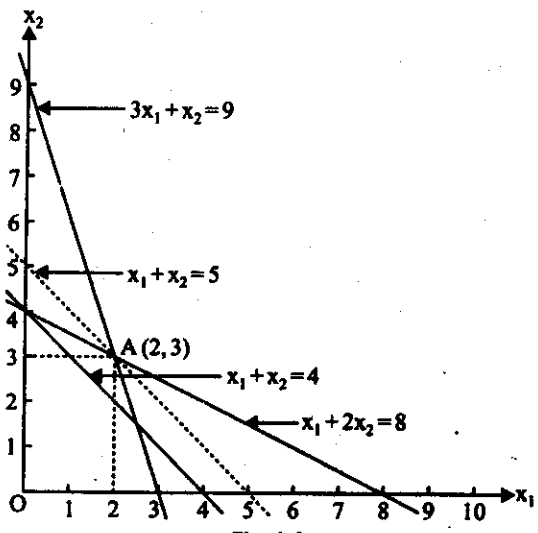


Fig. 1.4
Simultaneous Equations

In this case, any two of the three equations would have provided the unique solution point A, so one of the equations is said to be ***redundant (in excess)***.

Again, let us change the problem. We consider the single equation : $x_1 + 2x_2 = 8 \qquad \ldots(1\cdot4)$

Of course, there exist values of x_1 and x_2 satisfying the equation (1·4). But, besides the solution $x_1 = 2, x_2 = 3$ all points on the line $x_1 + 2x_2 = 8$ satisfy this equation. So the number of solutions is infinite. Although, some specific solutions can be obtained by setting one of the variables equal to some value. For example, set $x_2 = 0$, then find $x_1 = 8$. We will find some such solutions particularly useful in linear programming (called *basic solutions* of linear programming problem.)

Summary of results for two-variable case :

We may summarize the results obtained for the simultaneous equations of two variables :

(i) Two variables and more than two equations normally have no solution.

(ii) Two variables and two equations normally have a unique solution.

(iii) Two variables and less than two equations normally have an infinite number of solutions.

(iv) Redundant equations may reduce case (1·1) to case (1·2) or (1·3), and may reduce the case (1·2) to case (1·3).

(v) Inconsistent equations in case (1·2) have no solution.

The results (iv) and (v) generalize to the n-variable case as follows :

(a) n–variables and more than n equations have no solution, unless there are redundant equations (in which case the solution may be unique, or there may be an infinite number of solutions).

(b) n variables and n equations have a unique solution, unless the equations are inconsistent (there is no solution) or unless some equations are redundant (the number of solutions is infinite.)

(c) n variables and less than n equations have an infinite number of solutions, unless there are inconsistent equations (in which case there is no solution).

1.6 NUMERICAL SOLUTION OF SIMULTANEOUS EQUATIONS

For simplicity, again consider the system (1·1) of simultaneous equations :

$$x_1 + 2x_2 = 8$$
$$3x_1 + x_2 = 9.$$

In matrix notation, we can write as

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \quad \text{or} \quad \mathbf{Ax} = \mathbf{b}, \qquad \text{...(1·5)}$$

where $$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{and } \mathbf{b} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}.$$

Now using the fact that $A^{-1}A = I$ and $I\mathbf{x} = \mathbf{x}$, we premultiply both sides of (1·5) by A^{-1} and simplify :

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \text{ or } \mathbf{x} = A^{-1}\mathbf{b}.$$

Thus, formally, the solution to (1·5) can be obtained if we know the inverse of matrix A.

However, this is not an efficient method of solving the equations because, large number of operations are required to find the inverse of A. So the method based on the technique of *Gaussian elimination* require fewer operations.

1.6–1 Gauss-Jorden Method

The solution of simultaneous linear equations can be easily obtained by simple version of Gauss-Jordan method. The step by step procedure is outlined below :

Step 1. The first step is to form the augmented matrix [A : **b**] in which the column vector **b** now forms the additional column.

Step 2. Then we apply either the following row operations systematically, to obtain a matrix with ones and zeros in appropriate positions :

(i) multiply or divide all elements of a row by some suitable number; or

(ii) replace a row by the sum of that row and a multiple of some other row.

Step 3. Finally, the solution to the original system may be read off.

Following numerical examples will make the procedure clear.

Example 1. *Obtain the solution of the system of simultaneous equations :* $x_1 + 2x_2 = 8,\ 3x_1 + x_2 = 9$, *by Gauss-Jordan method.*

Solution :

Step 1. The augmented matrix is obtained as follows : $[A : \mathbf{b}] = \begin{bmatrix} 1 & 2 & : & 8 \\ 3 & 1 & : & 9 \end{bmatrix}$

Step 2. *(i)* Subtract 3-times of first row from second row to get, $\begin{bmatrix} 1 & 2 & : & 8 \\ 0 & -5 & : & -15 \end{bmatrix}$

(ii) Divide third row by -5 to get, $\begin{bmatrix} 1 & 2 & : & 8 \\ 0 & 1 & : & 3 \end{bmatrix}$

(iii) Subtract 2–times of second row from first row to get, $\begin{bmatrix} 1 & 0 & : & 2 \\ 0 & 1 & : & 3 \end{bmatrix}$

Step 3. Rewriting the set of equations, we have $\begin{Bmatrix} x_1 + 0x_2 = 2 \\ 0x_1 + x_2 = 3 \end{Bmatrix}$ which immediately give the solution $x_1 = 2, x_2 = 3$.

Example 2. *Solve the system of equations :*

$$\begin{aligned} x_1 + 2x_2 + \quad\quad x_4 \quad\quad &= 7, \\ x_2 + x_3 + x_4 + x_5 &= 24, \\ x_1 \quad - 3x_3 \quad + 2x_5 &= 8, \end{aligned}$$

for x_1, x_3 *and* x_5 *in terms of the remaining variables* x_2 *and* x_4.

Solution. The given system of equations can be written in the matrix form as :

$$\begin{matrix} & C_1 & C_2 & C_3 & C_4 & C_5 & & C_6 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ R_1 \rightarrow & 1 & 2 & 0 & 1 & 0 & : & 7 \\ R_2 \rightarrow & 0 & 1 & 1 & 1 & 1 & : & 24 \\ R_3 \rightarrow & 1 & 0 & -3 & 0 & 2 & : & 8 \end{matrix}$$

Since the columns corresponding to variables x_1, x_3, and x_5 are indicated by C_1, C_3 and C_5 respectively, we apply operation *(i)* or *(ii)* to rows R_1, R_2 and R_3 till the columns C_1, C_3 and C_5 becomes

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{and} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{respectively.}$$

Thus applying operation $R_3 - R_1$, we get $\begin{vmatrix} 1 & 2 & 0 & 1 & 0 & : & 7 \\ 0 & 1 & 1 & 1 & 1 & : & 24 \\ 0 & -2 & -3 & -1 & 2 & : & 1 \end{vmatrix}$

Applying $3R_2 + R_3$, we get $\begin{vmatrix} 1 & 2 & 0 & 1 & 0 & : & 7 \\ 0 & 1 & 1 & 1 & 1 & : & 24 \\ 0 & 1 & 0 & 2 & 5 & : & 73 \end{vmatrix}$

Applying $1/5\ R_3$, we get $\begin{vmatrix} 1 & 2 & 0 & 1 & 0 & : & 7 \\ 0 & 1 & 1 & 1 & 1 & : & 24 \\ 0 & 1/5 & 0 & 2/5 & 1 & : & 73/5 \end{vmatrix}$

Applying $R_2 - R_3$, we get $\begin{vmatrix} 1 & 2 & 0 & 1 & 0 & : & 7 \\ 0 & 4/5 & 1 & 3/5 & 0 & : & 47/5 \\ 0 & 1/5 & 0 & 2/5 & 1 & : & 73/5 \end{vmatrix}$

Rewriting as a set of equations, we have

$$\begin{aligned} x_1 + 2x_2 \quad + x_4 \quad &= 7 \\ \tfrac{4}{5}x_2 + x_3 + \tfrac{3}{5}x_4 \quad &= 47/5, \\ \tfrac{1}{5}x_2 \quad + \tfrac{2}{5}x_4 + x_5 &= 73/5, \end{aligned}$$

which gives us the required solution :

$$x_1 = 7 - 2x_2 - x_4,\quad x_3 = \frac{47}{5} - \frac{4}{5}x_2 - \frac{3}{5}x_4,\quad x_5 = \frac{73}{5} - \frac{1}{5}x_2 - \frac{2}{5}x_4.$$

1.6–2 Computer Program for Gauss–Jorden Method

A FORTRAN subroutine for solving a system of equations by the Gauss-Jordan method is given below.

```
      SUBROUTINE GJ (A, N)
      DIMENSION A (20, 21)
      N1 = N + 1
      DO 1K = 1 , N
      K1 = K + 1  1
      DO 2J = K1 , N1
2.    A (K , J) = A (K , J)/A (K , K)
      DO 1I = 1 , N
      IF (I . EQ . K) GO TO 1
      DO 3J = K1 , N1
3     A(I , J) = A (I , J) – A(1 , K) * A(K , J)
1     CONTINUE
      RETURN
      END
```

III – Calculus of Finite Differences

1.7 DIFFERENCE OPERATOR

Let us consider a function $y = f(x)$ defined only for integral values of independent variable x. Here y is a dependent variable.

Suppose we are given equidistant values (finite in number) a , $a + h$, $a + 2h$, $a + 3h$, ... for variable x at an interval of 'h'. Then, the corresponding values of the variable y are :

$$f(a) , f(a+h) , f(a+2h) , \ldots , \text{ and so on.}$$

The values of independent variable x are known as *arguments* and the corresponding values of the dependent variable y are called *entries, e.g.* $f(a+h)$ is the entry corresponding to the argument $a + h$, and so on. The interval 'h' is called the *difference interval.* Unless otherwise stated, the interval of difference is always taken as unity, *i.e.* $h = 1$.

Now we shall define a difference operator Δ which is of fundamental importance in the calculus of finite differences.

Definition. The 'first difference' of $f(x)$ is denoted by $\Delta f(x)$ defined by the formula

$$\Delta f(x) = f(x+1) - f(x) \text{ (taken } h = 1 \text{ , here).} \qquad \ldots(1{\cdot}6a)$$

Successive differences can also be defined. For example,

$$\Delta^2 f(x) = \Delta\,[f(x+1) - f(x)] = f(x+2) - 2f(x+1) + f(x) . \qquad \ldots(1{\cdot}6b)$$

We call Δ^2 the *second-order difference operator* or difference operator of order 2. In general, we define the $(n+1)th$ order difference operator by,

$$\Delta^{n+1} f(x) = \Delta\,[\Delta^n f(x)] \text{ , for } n = 0 , 1 , 2 , 3 , \ldots \qquad \ldots(1{\cdot}7a)$$

Further, it can be easily verified that

$$\Delta^{n+1} f(x) = \Delta^n\,[\Delta f(x)] = \Delta^n\,[f(x+1) - f(x)] = \Delta^n f(x+1) - \Delta^n f(x). \qquad \ldots(1{\cdot}7b)$$

Now, we observe that $(1{\cdot}7b)$ is true for all integral values, $n \geq 0$, if

$$\Delta^{\circ} \equiv \text{Identity operator, e.g. } \Delta^{\circ} f(x) = f(x) : \text{ and } \Delta^1 \equiv \Delta.$$

1.8 FACTORIAL NOTATION

Definition. The nth factorial of x, denoted by $x^{(n)}$, is defined by

$$x^{(n)} = x(x-1)\,(x-2)\,\ldots\,(x-n+1) ,$$

where by convention $x^{(0)} = 1$.

Similarly, the nth negative factorial of x is defined by $x^{(-n)} = \dfrac{1}{x\,(x+1)\,(x+2)\,\ldots\,(x+n-1)}$.

1.9 FORMULAE FOR FIRST DIFFERENCE OF COMBINATION OF TWO FUNCTIONS

If $f(x)$ and $g(x)$ are any two functions of x, then the following formulae can be readily verified.

1.	$\Delta[f(x)+g(x)]=\Delta f(x)+\Delta g(x)$	6.	$\Delta x^{(n)}=nx^{(n-1)}$, where $x^{(n)}$ is the *factorial function of degree n.*
2.	$\Delta[\alpha f(x)]=\alpha\,\Delta f(x)$, where α is some constant.	7.	In particular, $\Delta x=(x+1)-x=1$ or $1=\Delta x$.
3.	$\Delta[f(x)\,g(x)]=f(x)\,\Delta g(x)+g(x+1)\,\Delta f(x)$ $=g(x)\,\Delta f(x)+f(x+1)\,\Delta g(x)$ $=f(x)\,\Delta g(x)+g(x)\,\Delta f(x)+\Delta f(x)\,\Delta g(x)$	8.	$\Delta[a^x]=a^x(a-1)$, where a is some constant.
4.	$\Delta\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)\,\Delta f(x)-f(x)\,\Delta g(x)}{g(x+1)\,g(x)}$	9.	$\Delta[{}^xC_r]={}^xC_{r-1}$, where r is fixed and ${}^xC_r=\dfrac{x!}{r!\,(x-r)!}$.
5.	$\Delta\left[\dfrac{1}{f(x)}\right]=\dfrac{-\Delta f(x)}{f(x+1)\,f(x)}$		

[**Note.** It is observed that formula (6) is analogous to the formula $D(x^n)=nx^{n-1}$ in differential calculus.
Most of the above formulae are analogous to the corresponding formulae in differential calculus. Hence, these can be remembered easily.]

1.10 CONDITIONS FOR A MINIMUM (MAXIMUM) OF $f(x)$

The function $f(x)$ will have a '*local minimum*' at $x=x_0$, provided both the following conditions are satisfied :

$$f(x_0+1)-f(x_0)>0\,,\ i.e.\ \ \Delta f(x_0)>0,$$
$$f(x_0)-f(x_0-1)<0,\ i.e.\ \Delta f(x_0-1)<0\,.$$

Thus, we conclude that $f(x)$ will have a 'local minimum' at x_0 if

$$\Delta f(x_0-1)<0<\Delta f(x_0)\,. \qquad ...(1{\cdot}8)$$

The function $f(x)$ is said to have *'absolute minimum'* at x_0 if $f(x_0)\le f(x)$ for all x.

Therefore, sufficient conditions for $f(x)$ to have an ***'absolute minimum'*** at x_0 are :

$$\Delta f(x_0-1)<0<\Delta f(x_0)\,,\ \text{and also that } \Delta^2 f(x)\ge 0 \text{ for all } x\,.$$

The sufficient conditions for $f(x)$ to have an ***absolute maximum*** at x_0 are analogous, *i.e.* $\Delta f(x_0-1)>0>\Delta f(x_0)$, and also that $\Delta^2 f(x)\le 0$ for all x.

Remark. It is to be noted that these conditions are sufficient for a minimum (maximum), however, they are not necessary. A precise statement of necessary condition is not given here.

1.11 SUMMATION OF SERIES

If $f(x)$ be the continuous function defined for $a\le x\le b$, by fundamental theorem of integral calculus, we know the formula

$$\int_a^b f(x)\,dx=[F(x)]_a^b=F(b)-F(a)\,, \qquad ...(1{\cdot}9)$$

where $F(x)$ is an anti-derivative of $f(x)$.

Now the following theorem will give us an expression if $f(x)$ is defined only for the integral values of independent variable x ranging from a to b.

Theorem. *If $f(x)$ be defined only for integral values of independent variable x,* then

$$\sum_{x=a}^{b} f(x)=f(a)+f(a+1)+\ldots.+f(b)=[F(x)]_a^{b+1}=F(b+1)-F(a)\,,$$

where $F(x)$ is an anti-difference (instead of anti-derivative) of $f(x)$, *i.e.* $\Delta F(x)=f(x)$.

Proof. Since $\Delta F(x)=f(x)$, we have

$$\sum_{x=a}^{b} f(x)=\sum_{x=a}^{b}\Delta F(x)=\sum_{x=a}^{b}[F(x+1)-F(x)]$$
$$=[F(b+1)-F(b)]+[F(b)-F(b-1)]+\ldots\ldots+[F(a+1)-F(a)]$$
$$(\text{since } x=b\,,\,b-1\,,\,b-2\,,\,\ldots\,,\,a+1\,,\,a)$$
$$=F(b+1)-F(a)\,.$$

Thus, if $F(x)$ is an anti-difference of $f(x)$, *i.e.* $\Delta^{-1} f(x)=F(x)$, then

$$\sum_{x=a}^{b} f(x) = [F(x)]_a^{b+1} = F(b+1) - F(a). \quad \text{...(1·10)}$$

This completes the proof of the theorem.

Note. We note that the upper limit in expression (1·10) is $b+1$ and not b, unlike in expression (1·9).

Example 3. *Use the method of finite differences to find the sum to n terms of the series whose x-th term is given by*

$$f(x) = \frac{x+3}{x(x+1)(x+2)}.$$

Solution. We can express

$$f(x) = (x+3)/x(x+1)(x+2) = [(x+1)(x+2)]^{-1} + 3[x(x+1)(x+2)]^{-1} = x^{(-2)} + 3(x-1)^{(-3)},$$

where $x^{(-r)}$ represents the negative factorial function of x.

Let $F(x) = \Delta^{-1} f(x) = \Delta^{-1}[x^{(-2)} + 3(x-1)^{(-3)}] = \frac{x^{(-1)}}{-1} + \frac{3(x-1)^{(-2)}}{-2}$

Hence by virtue of result (2·20), we have

$$\sum_{x=1}^{n} f(x) = [F(x)]_1^{n+1} = F(n+1) - F(1)$$

$$= \left[-(n+1)^{(-1)} - \tfrac{3}{2} n^{(-2)}\right] - \left[-1^{(-1)} - \tfrac{3}{2} 0^{(-2)}\right]$$

$$= -\left[\frac{1}{n+2} + \frac{3}{2(n+1)(n+2)}\right] + \left[\frac{1}{2} + \frac{3}{2 \times 1 \times 2}\right] = \frac{5n^2 + 11n}{4(n+1)(n+2)}. \quad \textbf{Ans.}$$

1.12 SUMMATION BY PARTS

As we know from integral calculus, a definite integral of the form $\int_a^b f(x)\, dg(x)$ can be easily evaluated by integrating by parts, as follows :

$$\int_a^b \overset{\text{I}}{f(x)}\, \overset{\text{II}}{dg(x)} = \left[f(x)\, g(x)\right]_a^b - \int_a^b g(x)\, df(x) \quad \text{...(1·11)}$$

$$= [f(b)\, g(b) - f(a)\, g(a)] - \int_a^b g(x)\, df(x),$$

where, $f(x)$ and $g(x)$ are two continuous functions of x defined for $a \le x \le b$.

Now, if $f(x)$ and $g(x)$ are defined only for integral values of x, ranging from a to b, we have a formula nearly analogous to (1·11) in finite differences, which follows :

$$\sum_{x=a}^{b} \overset{\text{I}}{f(x)}\, \Delta \overset{\text{II}}{g(x)} = [f(x)\, g(x)]_a^{b+1} - \sum_{x=a}^{b} g(x+1)\, \Delta f(x) \quad \text{...(1·12)}$$

Example 5. *Evaluate :* $\sum_{x=1}^{k} xa^x$.

Solution. First, we express $\sum_{x=1}^{k} xa^x$ in the form of *l.h.s.* of above formula (1·12). Thus, we have

$$\sum_{x=1}^{k} xa^x = \sum_{x=1}^{k} \overset{\text{I}}{x}\, \Delta\, \overset{\text{II}}{(a^x/(a-1))} \quad [\text{because } \Delta a^x = a^x(a-1)]$$

Hence, applying the above formula (1·12), we have

$$\sum_{x=1}^{k} xa^x = \left[x \cdot \frac{a^x}{a-1}\right]_1^{k+1} - \sum_{x=1}^{k} \left(\frac{a^{x+1}}{a-1}\right) \Delta x$$

$$=\left(\frac{(k+1)\,a^{k+1}}{a-1}-\frac{a}{a-1}\right)-\frac{1}{a-1}\sum_{x=1}^{k}\Delta\left(\frac{a^{x+1}}{a-1}\right)1\,.$$

[because $\Delta a^{x+1}=a^{x+1}(a-1)$ and $\Delta x=1$].

$$=\frac{(k+1)\,a^{k+1}}{a-1}-\frac{a}{a-1}-\frac{1}{(a-1)^2}\Big[\,a^{x+1}\Big]_1^{k+1}$$

[using the result (1·10)]

$$=\frac{(k+1)\,a^{k+1}}{a-1}-\frac{a}{a-1}-\frac{1}{(a-1)^2}\times[a^{k+2}-a^2]\,.$$

***

2

Review Of Probability And Statistics

2.1 INTRODUCTION

So far we were concerned with the problems in which distances, times and other variables were almost always given precise numerical values. In many situations, however, there are quantities which are subject to random variation, like monthly level of sales in a supermarket shop. Since it is not always necessary to construct a ***deterministic*** model, we can choose to adopt a ***stochastic*** model. In order to apply the optimization principles of Operations Research to systems whose details we do not fully understand or cannot easily analyse, we borrow ideas from statistics and build-up stochastic models.

In this chapter, we develop the concept of probability because this is one of the main statistical tools used in operations research.

The student who feels confident of his abilities in the area of probability theory may skin, or skip altogether, the entire chapter. But, the student who has no previous knowledge to the subject is well advised to go through this chapter carefully.

2.2 CONCEPT OF UNCERTAINTY

Consider the following experiment.

A person X tosses a coin and Y also tosses a similar coin simultaneously and observe the result.

It is *uncertain* to say :

(*i*) Whether both heads come up or (*ii*) both tails come up or (*iii*) one head and one tail come up.

The situation of uncertainty can also be observed by considering two sets A and B in three different situations as shown in the followng diagram :

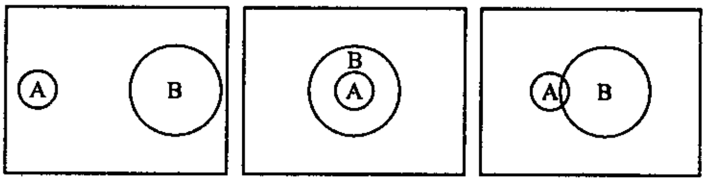


Fig. 2.1.

In situations (*i*) and (*ii*), it is certain that $x \in A \Rightarrow x \notin B$; $x \in A \Rightarrow x \in B$, respectively; while in situation (*iii*) if $x \in A$ we cannot say whether the element $x \in B$ or $x \notin B$ which is the situation of uncertainty.

Before giving a formal definition of probability, it is necessary to introduce the concepts of *probability, sample space and events.*

2.3 CONCEPT OF PROBABILITY

In our daily life, we often come across sentences like :

(*i*) It is impossible that he may refuse to do my work. (*ii*) Probability, it will rain tomorrow.

Here, in sentence (*i*), we observe that the probability of doing work is unity and, therefore, probability of refusing to do work will be zero. Consequently, in sentence (*ii*) the probability of raining lies between *zero* and *unity.*

2.4 SAMPLE SPACE

A sample space can be defined as the set of all possible outcomes of an experiment and is denoted by S.

Suppose all possible outcomes of an experiment are denoted by $e_1, e_2, e_3, \ldots e_n$, which are such that no two or more of them can occur simultaneously, and exactly one of the outcomes $e_1, e_2, e_3, \ldots, e_n$ must occur whenever the experiment is performed.

Definition. *The set* $S = \{e_1, e_2, e_3, \ldots, e_n\}$ *is called a* ***sample space*** *of an experiment satisfying the following two conditions :*

(*i*) *Each element of the set S denotes one of the possible outcomes.*

(*ii*) *The outcome is one and only one element of the set S whenever the experiment is performed, e.g., in tossing a coin, sample space consists of head or tail,* i.e. $S = \{H, T\}$;
when two coins are tossed simultaneously, $S = \{HH, HT, TH, TT\}$;
when three coins are tossed simultaneously, $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$.

For example, we perform an experiment of rolling a six-faced die whose faces are numbered as e_1, e_2, e_3, e_4, e_5 and e_6. Whenever the die comes to rest, one of the numbers e_1, e_2, e_3, e_4, e_5, or e_6 appears on the top face. If we ask the question, "Which number came out on the top" ? there are six possible answers. Each of these possible answers is a possible outcome of the experiment. The set $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ whose every member designates one of these possible outcomes, is called a ***Sample Space*** for the experiment.

Instead of asking, "Which number come out on the top" ? we might ask, "Is the number that comes out on top odd or even" ? Then the set of words {odd, even}, designating two possible outcomes, is also a sample space.

Example 1. *From an urn containing 4 balls of different colours, i.e., Red (R), Blue (B), Yellow (Y) and Green (G), two balls are drawn; (i) simultaneously (ii) one after the other with replacement. Define the sample space in both the cases.*

Solution. (*i*) The number of ways of drawing 2 balls out of 4 balls = ${}^4C_2 = 6$.

The sample space is the set of all possible combinations of two balls of different colour.

Hence, the sample space in first case is : $S = \{BR, YR, GR, BY, BG, GY\}$. **Ans.**

(*ii*) In this case, the balls are drawn one after the other with replacement. Therefore, the combination of the same colour balls is also possible.

Hence, the sample space is given by the set

$$S = \{GG, GY, GB, GR, YR, YB, YY, YG, BB, BR, BY, BG, RR, RB, RY, RG\}.$$ **Ans.**

2.5 ELEMENTARY EVENTS

Definition. *The element or the point of a sample space associated with an experiment are called the elementary events of the experiment.*

It is the obvious fact that no two elementary events can occur simultaneously, and exactly one of these must occur in a single trial. For example, a marksman cannot hit and miss the target simultaneously, and further a single shot must either hit the target or miss it.

2.6 ACCEPTABLE ASSIGNMENT OF PROBABILITIES TO ELEMENTARY EVENTS

Consider a sample space S of an experiment consisting of n elementary events $e_1, e_2, e_3, \ldots, e_n$.

Definition. *If to each elementary event* $e_i \in S, i = 1, 2, \ldots, n$, *we assign a real number* $P(e_i)$ *called the* ***probability*** *of an elementary event* e_i *such that :*

(*i*) *the probability of each* $e_i (i = 1, 2, \ldots, n)$ *is a non-negative real number, i.e.,* $P(e_i) \geq 0$ for $i = 1, 2, \ldots, n$;

(*ii*) *the sum of the probabilities assigned to all elementary events of the sample space S is unity, i.e.,*

$$P(e_1) + P(e_2) + \ldots + P(e_n) = 1;$$

then such an assignment of probabilities to elementary events is called the acceptable assignment of probabilities to elementary events.

For example, in a coin tossing experiment, the sample space is $S = \{H, T\}$ where H and T are used to stand for *Head* and *Tail*, respectively.

It is evident that

(*i*) If $P(H) = p$ (say) and $P(T) = 1 - p$ where $0 \leq p \leq 1$, then it is an acceptable assignment of probabilities.

(*ii*) $P(H) = -\frac{1}{2}$ and $P(T) = \frac{3}{2}$ is not an acceptable assignment because probability cannot be less than zero and greater than unity although the sum $P(H) + P(T) = 1$.

(*iii*) $P(H) = 0.50$ and $P(T) = 0.49$ is also not an acceptable assignment because $P(H) + P(T) \neq 1$.

2.7 NATURAL ASSIGNMENT OF PROBABILITIES TO ELEMENTARY EVENTS

If each elementary event is assigned the same probability, *i.e.*,

$$P(e_1) = P(e_2) = P(e_3) = \ldots = P(e_n) = 1/n, \text{ [because } P(e_1) + P(e_2) + \ldots + P(e_n) = 1]$$

then such an assignment is called ***natural assignment.***

For example, in a coin tossing experiment $S = \{H, T\}$, $P(H) = P(T) = \frac{1}{2}$ satisfying $P(H) + P(T) = 1$ is the natural assignment.

2.8 EVENTS

Consider, for example, the experiment of throwing a six-faced die. The sample space is $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, where the appearance of face numbered 1 is denoted by e_1, face numbered 2 is denoted by e_2 etc. If we are interested in the event of appearing even numbered faces up, then the event will have the outcomes e_2, e_4, e_6 only and hence this event is denoted by $E_1 = \{e_2, e_4, e_6\}$, which is the subset of S. We now watch for the event, "The face number that turns up is divisible by 3". This event will be denoted by $E_2 = \{e_3, e_6\}$ which is also a subset of S. Thus we may define the event as follows :

Definition 1. ***(Event)*** *Every subset of a sample space S of an experiment is called an **event** generally denoted by E.*

In particular, the sample space S itself is called the ***certain event*** which is obviously the subset of itself. And the impossible event will be denoted by the empty subset ϕ of sample space S.

Simple Event. Furthermore, among the subsets of S some subsets are containing only one member. Here is a complete list of these one-member subsets $\{e_1\}$, $\{e_2\}$, $\{e_3\}$, $\{e_4\}$, $\{e_5\}$, and $\{e_6\}$. Because the one-member events play an important role in the theory of probability, so we give them a special name.

Definition 2. ***(Simple Event).*** *Any event that contains only one member of a sample space is called a **simple event** in this space.*

For example, let a die be rolled once and A be the event that face 3 is turned up, then A is called a simple event.

Definition 3. ***(Compound Events).*** *When an event is decomposable into a number of simple events, then it is called a **compound event.***

For example, the event, "the sum of the two numbers shown by the upper faces of the two dice is seven in the simultaneous throw of the two unbiased dice", is a compound event as it can be decomposed into the following simple events (1, 6), (6, 1), (5, 2), (2, 5), (4, 3), (3, 4).

Definition 4. ***(Equally Likely Events).*** *Events are said to be equally likely if there is no reason to expect any one in preference to any other i.e., when the probability of happening of two or more events is the same, they are called **equally likely events.***

For example, if a coin is tossed then there may be either head up or tail up *i.e.*, there is no reason to expect the occurrence of tail in preference to head up.

Definition 5. ***(Mutually Exclusive Events).*** *Events are said to be **mutually exclusive** when the happening of one excludes the happening of the other i.e., no any two events can occur simultaneously.*

For example, when a coin is tossed, event of throwing head and event of throwing tail are mutually exclusive events.

Event of 'rolling a face numbered less than 3' and event of 'rolling even numbered face on a die' are not mutually exclusive.

Definition 6. ***(Dependent and Independent Events).*** *Two events are said to be **independent** when occurrence of one has no effect on the probability of other.*

*Events are said to be **dependent**, if they are not independent.*

Illustrative Examples

Example 2. *If a bag contains seven balls, and one ball is drawn from it and is replaced back, then a second ball is drawn from it; the probability of the second drawing is independent of the first and so the two drawings (events) will be independent. Again if first ball is not replaced back, then a second ball is drawn from it; the prob. of the second drawing is dependent of the first and so the two (events) are dependent.*

Example 3. *If a die is rolled twice, event of getting face number 5 in first tossing and event of getting face number 3 in the second tossing are independent.*

2.9 SET NOTATIONS FOR EVERYDAY LANGUAGE

$E \subset S$	:	*Event E.*
$e \in S$	:	e is an outcome of an experiment whose sample space is S.
$\bar{E}$	:	Complementary event of event E.
$e \in E$	:	Event E occurs.
$e \in \bar{E}$	:	Event E does not occur.
$e \in E \cup F$	:	Event E or even F occurs.
$e \in E \cap F$	:	Event E and event F occurs.
$E = \phi$	:	Event E is impossible.
$E = S$	:	Event E is a certain event.
$F \subset E$	:	If event E occurs, then event F will also occur.
$E \cap F = \phi$	:	If event E occurs, then event F does not occur, *i.e.*, E and F are independent events.

2.10 PROBABILITY OF AN EVENT

Definition. ***Probability.*** *With each event E_i in a finite sample space S, we associate a real number, say $P(E_i)$, called the probability of an event E_i satisfying the following conditions :*

(*i*) $0 \le P(E_i) \le 1$. *This implies that the probability of an event is always non-negative and can never exceed unity.*

(*ii*) $P(E_1 \cup E_2 \cup E_3 \ldots \cup E_k) = P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_k)$ *where* $E_1, E_2, E_3, \ldots, E_k$ *are mutually exclusive events in S.*

(*iii*) $P(S) = 1$, *where the event S is the entire sample space called the* ***certain event.***

Q. What are the axioms of probability ? [Kanpur 96]

2.11 FREQUENCY INTERPRETATION (Classical Definition of Probability)

If n denotes the number of times the experiment is performed and m the number of successful occurrences of the event E_i in the n trials, then $P(E_i)$ can be defined as

$$P(E_i) = \lim_{n \to \infty} \frac{m}{n} \quad \text{(finite and unique)},$$

provided the outcomes of event E_i are :

(*i*) **exhaustive :** which implies (as in a coin tossing experiment) that it must fall either head or tail but cannot stand on edge;

(*ii*) **mutually exclusive :** which implies that head and tail must not occur simultaneously. We also know the fact that a marksman cannot hit and miss the target simultaneously, and moreover a single shot must either hit the target or miss it; and

(*iii*) **equally likely :** which means that head and tail both have equal chances to occur.

Also, the ratio m/n does fulfil the following four conditions of probability :

(*a*) $0 \le m/n \le 1$.

(*b*) $$\frac{m_1 + m_2 + m_3 + \ldots + m_k}{n} = \frac{m_1}{n} + \frac{m_2}{n} + \frac{m_3}{n} + \ldots + \frac{m_k}{n},$$

where E_1, E_2, E_3, ..., E_k are independent (disjoint) events and each occurs m_1, m_2, m_3, ..., m_k number of times, respectively, in n number of trials.

(*c*) $n/n = 1$, since the event E_i must occur every time the experiment is performed, therefore $m = n$.

(*d*) $0/n = 0$, $m = 0$, since the event E_i cannot occur in any trial.

Actual probability of an event E_i cannot be determined by classical definition because of $n \to \infty$. Although it is possible to estimate the probability of an event experimentally by drawing a graph between n and $m/n \to \infty$, the ratio m/n becomes very nearly constant which will be the estimated probability of an event.

2.11-1 The Playing Card Example

Example 4. *A playing card is chosen at random from a pack of 52 cards. The 52 possible outcomes to this trial can be displayed by a diagram called the* ***sample space (or outcome space)*** *diagram.*

Sample Space Diagram

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Hearts	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥
Diamonds	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦
Clubs	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣
Spades	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠

Now the following questions may be answered.

Question 1. *What is the probability of choosing the 3 of Diamonds ?*

Answer. In view of the argument that for every 52 trials, one of the cards will be the 3 of *Diamonds*, we deduce that the *probability of the 3 of Diamonds*, which is written as **P(3 of diamonds)**, is $1/52$. It is worthnoting that while repeating a trial, the conditions which existed originally must be restored. In this case, the card chosen is considered to be returned back to the pack each time otherwise the trial is not an exact repetition of the previous one.

Question 2. *What is the probability of choosing any* 3 ?

Answer. We know that an average 4 out of every 52 cards chosen will be a '3' as there are four '3'*s* in the whole pack. So the answer will be $4/52$.

We can also deduce this answer from the above as the fraction of trials resulting in 'any 3' is the sum of the fractions of times the 3 of *Hearts*, the 3 of *Diamonds*, the 3 of *Clubs* and the 3 of *Spades* occur. Hence we have

$$P(\textbf{any 3}) = \frac{1}{52} + \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{4}{52}, \quad \text{as before.} \qquad ...(2.1)$$

Question 3. *What is the probability of not getting a* 3 ?

Answer. Since there are 48 cards which are not '3'*s*, therefore, P (not 3) will be $48/52$.

Alternatively, it can be argued that on any trial the card chosen is either a '3' or not a '3'. So the fraction of trials on which a '3' occurs and the fraction on which a '3' does not occur must add up to 1.

$$P(\textbf{not 3}) = 1 - P(\text{any 3}) = 1 - \frac{4}{52} = \frac{48}{52}, \text{ as before.} \qquad ...(2.2)$$

In general, if the letter A stands for any event, then

$$P(\textbf{not } A) = 1 - P(A) \qquad ...(2.3)$$

Question 4. *What is the probability of choosing any 3 or any Diamond or both ?*

Solution. It can be seen from the following diagram that there are 16 playing cards which are either a '3' or a Diamond, or Both. Therefore, P (3 or Diamond or Both) = $16/52$.

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Hears	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥
Diamonds	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦
Clubs	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣
Spades	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠

Alternatively, the answer can also be obtained by as well as simply counting the dots in the above diagram. The set 'any 3 or any Diamond or Both' is the union of the sets 'any 3 which contains 4 cards, and 'any Diamond', which contains 13 cards. The number of cards in their union is equal to the sum of these numbers minus the number of cards in the space where they overlap. Any points in this space, called the *intersection* of the two sets, is counted here twice, once in each set. Dividing all these numbers (4, 13 and 1) by 52 to change them into probabilities, we observe that—

P(any 3 or any Diamond or Both) = P(any 3) + P(any Diamond) – P(3 of Diamond)

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}, \text{ as we obtained earlier.} \quad ...(2.4)$$

In general, if the letters A and B stands for any two events, then above equation can be written as

$$P(A \text{ or } B \text{ or Both}) = P(A) + P(B) - P(A \cap B). \quad ...(2.5)$$

Question 5. *What is the probability of choosing any* 3 *or say* 7 ?

Answer. The number of cards which are any '3 or any 7' is 8 and therefore the desired answer is 8/52.

Alternatively, using the above formula with A corresponding to the event 'any 3' and B to the event 'any 7' we get.

P(any 3 or any 7 or Both) = P (any 3) + P (any 7) – P (any 3 and any 7)

$$= \frac{4}{52} + \frac{4}{52} - \frac{0}{52} = \frac{8}{52}. \quad ...(2.6)$$

Remember that P(any 3 and any 7) is zero because a selected card cannot be both a '3' and a '7'. So the events which cannot occur simultaneously like this are called mutually exclusive and the formula (2.5) will then reduce to a simple addition rule for their probabilities. An example of this can be seen in the answer of question 2.

Remember. 1. For any event A, $0 \le P(A) \le 1$, *i.e.*, the scale of probability extends from 0 to 1.

2. If A is an ***impossible*** event, then $P(A) = 0$.

3. If A is a ***certain*** event, then $P(A) = 1$.

Odds in favour or against an event. Instead of saying that the probability of the happening of an event is $\frac{m}{m+n}$, it is sometimes stated that odds are $m : n$ in favour of the event or $n : m$ against the event.

2.12 THE ADDITION LAW OF PROBABILITY

If the events $E_1, E_2, E_3, ..., E_n$ are mutually exclusive in pairs, then

$$P(E_1 \cup E_2 \cup E_3 \ldots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_n)$$

Theorem. *If $p_1, p_2, ..., p_n$ be separate probabilities of n mutually exclusive events, then the probability of the happening of any one of these events is given by $p = p_1 + p_2 + p_3 + \ldots + p_n$.*

Proof. Let $A_1, A_2, \ldots, A_n$ be n mutually exclusive events.

Let N be the number of cases which are equally likely, mutually exclusive and exhaustive.

Out of these N cases, let

No. of cases favourable to event $A_1 = m_1$

No. of cases favourable to event $A_2 = m_2$

... ...

No. of cases favourable to event $A_n = m_n$.

But $A_1, A_2, ..., A_n$ are mutually exclusive. Therefore, $m_1, m_2, ..., m_n$ are distinct and non over-lapping.

∴ Total number of cases which are favourable to either A_1, or A_2 ... or A_n are $= m_1 + m_2 + \ldots + m_n$

∴ The probability 'p' of any one of the events happening $= \dfrac{m_1 + m_2 + \ldots + m_n}{N}$

$$= \frac{m_1}{N} + \frac{m_2}{N} + \ldots + \frac{m_n}{N} = P(A_1) + P(A_2) + \ldots + P(A_n) \text{ [by def.]}$$

∴ $p = p_1 + p_2 + \ldots + p_n$ which proves the theorem.

Remember. Probability of occurrence of at least one of the two non-mutually exclusive events is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

2.13 THE CONDITIONAL PROBABILITY

Consider the two events E_1 and E_2 in a sample space S. Here E_1 represents the event that has occurred m_1 number of times in n (large) number of trials, and E_2 represents the event that has occurred m_2 number of times out of these m_1 number of occurrences of E_1. Therefore, by using the classical definition of probability, we can find the probability of the combined happening of E_1 and E_2 in the same trial as

$$P(E_1 \cap E_2) = \frac{m_2}{n} = \frac{m_1}{n} \cdot \frac{m_2}{m_1}. \quad ...(2.7)$$

Obviously, $m_1/n = P(E_1)$, but the relative frequency m_2/m_1 can approximately be taken as the *conditional probability of occurrence* of event E_2 given that E_1 has occurred $[P(E_1) \neq 0]$, which is denoted by $P(E_2 \mid E_1)$.

Now, (2.7) becomes

$$P(E_1 \cap E_2) = P(E_1)\, P(E_2 \mid E_1) \quad ...(2.8)$$

That is, *the probability that both E_1 and E_2 occur is equal to the prob. that E_1 occurs m_1 times the prob. that E_2 occurs given that E_1 has occurred.*

The above discussion suggests that we define the *conditional* probability of E_2 given E_1,

$$P(E_2 \mid E_1) = P(E_1 \cap E_2)/P(E_1), \text{ where } P(E_1) > 0. \quad ...(2.9)$$

Similarly, we can get

$$P(E_1 \mid E_2) = P(E_1 \cap E_2)/P(E_2), \text{ where } P(E_2) > 0. \quad ...(2.10)$$

Furthermore, $P(E_2 \mid E_1)$, the conditional probability of E_2 satisfies the following four properties.

P₁. $0 \leq P(E_2 \mid E_1) \leq 1$.

P₂. If E_2 is an event which cannot occur, then $P(E_2 \mid E_1) = 0$.

P₃. If the event E_2 is the entire sample space S, then $P(S \mid E_1) = 1$.

P₄. If E_2 and E_3 are two independent events in S, then $P(E_2 \cap E_3 \mid E_1) = P(E_2 \mid E_1) + P(E_3 \mid E_1)$.

In case the occurrence of E_1 does not affect the occurrence of E_2, we have

$$P(E_2 \mid E_1) = P(E_2). \quad ...(2.11)$$

Thus, E_1 and E_2 are independent if and only if,

$$P(E_1 \cap E_2) = P(E_1)\, P(E_2). \quad \text{[using results of (2.8) and (2.11) here]}$$

We now state an important law of probability :

The law of total probability. For any n events $E_1, E_2, ..., E_n$.

$$P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n) = P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_1E_2) \cdot P(E_n \mid E_1E_2 \ldots E_{n-1}).$$

If the events $E_1, E_2, ..., E_n$ are independent, then

$$P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n) = P(E_1).P(E_2) \ldots P(E_n)$$

Q. 1. Define conditional probability, and outline its properties.

2. A and B are any two events and the probability $P(B) \neq 1$, prove that $P(A \mid \bar{B}) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$. [IAS (Main) 97]

2.14 DISCRETE AND CONTINUOUS VARIABLES

Suppose we collect data for the size of families in a certain town. It is obvious that the number of members in each family would be in whole numbers (irrespective of age or sex). Thus, for example, there would be no family with 1.5 members. Such type of variable (number of family members in this example) is called a *discrete variable*. The school enrolment figures, number of passengers, etc., are other examples of it.

Having defined the discrete variable, we must next define the continuous type of variable. Suppose we are interested in measuring the heights of a large number of plants and if our unit of measurement is very fine, there would be no point along the scale of measurement (between the extreme values of the heights) at which we may not find the height of plant, no matter how finely we divide the scale. *A variable (the height in this example) which takes all possible values between its limits, say a and b, is known as a continuous variable.* Weights of school children, barometric pressures, temperatures etc., are other examples of it.

2.15 RANDOM VARIABLE OR CHANCE VARIABLE [IGNOU 98]

To define a *random variable*, we consider some experiment whose sample space is $S = \{e_1, e_2, e_3, \ldots, e_n\}$.

Definition. *A random variable X is a rule which associates uniquely a real number with every elementary event $e_i \in S$, $i = 1, 2, 3, \ldots, n$, i.e.,* ***a random variable is a real valued function which maps the sample space on to the real line.***

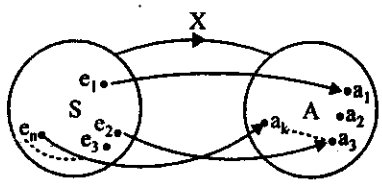


Fig. 2.2.

In other words, *a random variable is a real valued function defined over the sample space of an experiment i.e. a variable whose value is a number determined by the sample point (out-come of the experiment) of a sample space is called a random variable.*

For example, let X be a random variable which is the number of heads obtained in two independent tosses of a fair coin. Here $S = \{HH, HT, TH, TT\}$.

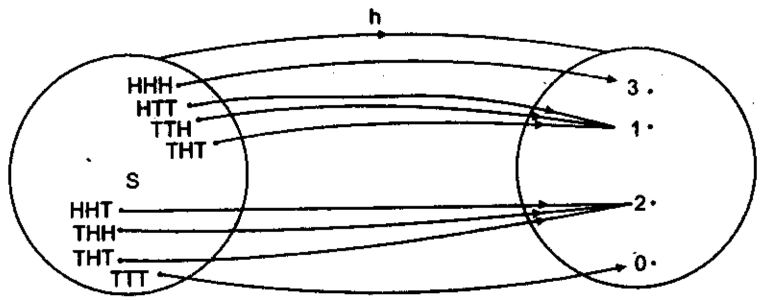
Then, $X(HH) = 2$, $X(HT) = 1$, $X(TH) = 1$, $X(TT) = 0$. Therefore, X can take values 0, 1, 2.

A random variable is also known as *stochastic variable.*

As another example, let us consider a three coin throwing experiment. If we write 'H' for turning head up and 'T' for turning tail up, obviously the sample space for this experiment will be

$$S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}.$$

Now, suppose that the underlying rule **h** is to *count the number of 'heads' that can turn up.* Then **h** will be a *random variable* which associates uniquely a real number with every elementary event of S as shown in the Venn diagram (see Fig. 2.3).



Domain of h is the set S. Range of h is the subset (0, 1, 2, 3) of real numbers.

Fig. 2.3.

Similarly, we can demonstrate the random variable **t** which is a rule : *to count the number of 'tails'* that can turn up in this experiment.

Furthermore, if the range of a random variable is a discrete variable, it is called a *discrete random variable.* In above example, **h** and **t** are discrete type of random variables. On the other hand, the range of *continuous random variable* will be the set of continuous real numbers. For example, if the experiment is concerned with measuring failures of an electronic component, the outcomes in this case are given by the time-to-failure which may assume any non-negative real value. In this case, the random variable is continuous.

2.16 PROBABILITY DISTRIBUTION OF A RANDOM VARIABLE

Let a random variable X assume values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$. Then different values of a random variable together with their corresponding probabilities form a probability distribution.

A random variable is said to be ***discrete*** if it assumes only a finite or infinite but denumerable number of values.

A random variable is said to be ***continuous*** if it assumes any value in an interval.

2.16-1 Discrete Probability Distribution

We consider an experiment with sample sapce $S = \{e_1, e_2, ..., e_n\}$, and suppose that a random variable X has been defined on S by the formula $X(e_i) = x_i$, $i = 1, 2, 3, ..., n$, where x_i is the real number associated with the elementary event e_i by the random variable X. It should be noted that all x_i, $i = 1, 2, ..., n$ are not necessarily different.

Now, we proceed to distribute probabilities to each event $E(x) = \{e_i \in S : X(e_i) = x_i\}$, $i = 1, 2, 3, ..., n$ *i.e.*, a subset of sample space S whose elements are ***uniquely*** associated to some real number, say x, by random variable X. To do this, we again associate each real number $x_1, x_2, ..., x_r$ $(r \leq n)$ to other real numbers $P(x_1), P(x_2), ..., P(x_r)$ respectively. Then such function P is called the ***probability distribution function*** of random variable X and the numbers $P(x_1), P(x_2), ..., P(x_r)$ are called ***probabilities*** of events $E(x_1), E(x_2), ..., E(x_r)$ respectively, provided :

(*i*) $0 \leq P(x_j) \leq 1$, (*ii*) $\Sigma P(x_i) = 1$ (*i.e.* sum of all probabilities is unity), $j = 1, 2, 3, ..., r$ and $r \leq n$.

Thus the probability distribution for discrete random variable X can be more conveniently represented in the following tabular form :

Table 2.1.

x	x_1	x_2	...	x_r
$P(x)$	$P(x_1)$	$P(x_2)$	...	$P(x_r)$
$P\{E(x)\}$	$P\{E(x_1)\} = p_1$	$P\{E(x_2)\} = p_2$	...	$P\{E(x_r)\} = p_r$

The following illustrative example will make the ideas clear.

Example. *We consider a three coin tossing experiment whose sample space is :*

$$S = \{HHH, HHT, HTH, THH, TTH, TTT, THT, HTT\}.$$

The random variable ***h****, which counts the number of heads that turn up, has the range {0, 1, 2, 3}, and the four possible events are :*

(*i*) The event which no head turns up $= E(0) = \{TTT\}$

(*ii*) The event when only one head turns up $= E(1) = \{HTT, THT, TTH\}$

(*iii*) The event when only two heads turn up $= E(2) = \{HHT, HTH, THH\}$

(*iv*) The event when all the three heads turn up $= E(3) = \{HHH\}$

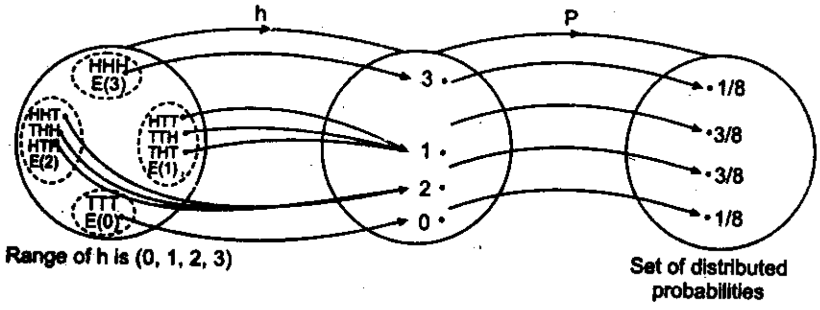


Fig. 2.4.

Thus the probability distribution is shown in Table 2.2 :

Table 2.2.

x	0	1	2	3
$P(x)$	1/8	3/8	3/8	1/8
$P\{E(x)\}$	$P\{E(0)\}$	$P\{E(1)\}$	$P\{E(2)\}$	$P\{E(3)\}$

2.16-2 Continuous Probability Distribution

Since X is a continuous random variable, it will have the infinite number of values in any class interval, however small. Thus, we can assign a probability to an interval of X.

The probability, that the continuous random variable X lies in the infinitesimal interval $(x, x + dx)$, can be interpreted as $f(x)\,dx$. Then $f(x)$ is known as the *probability density function* of random variable X.

The *distribution function, F(a),* usually be written as

$$F(a) = P\{X(e) \le a\} = \int_{-\infty}^{a} f(x)\,dx, \qquad \text{...(2.12)}$$

where $P\{X(e) \le a\}$ is the probability of an event whose, each outcome e is uniquely associated with all real numbers *not more than a* by continuous random variable X. Whenever there is no ambiguity, it can be written as $P(X \le a)$.

A knowledge of the density function enables us to calculate all sorts of probabilities.

$$P(a < X \le b) = F(b) - F(a) = \int_{a}^{b} f(x)\,dx.$$

Since a and b are two extreme values on the range of X with $a < b$, we can express $P(a < X \le b)$ in terms of *'distribution function'* provided the distribution function $F(x)$ is differentiable, *i.e.*,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\int_{a}^{b} f(x)\,dx\right]. \qquad \text{...(2.13)}$$

For $\{X \le b\}$ is the disjoint union of two events $\{X \le a\}$ and $\{a < X \le b\}$, hence

$$P\{X \le b\} = P\{X \le a\} + P\{a < X \le b\} \qquad \text{...(2.14)}$$

by using, $P(E \cup F) = P(E) + P(F)$, for disjoint events E and F.

Using (2.12), eqn. (2.14) can be written as :

$$F(b) = F(a) + P\{a < X \le b\}$$

or

$$P\{a < X \le b\} = F(b) - F(a) = [F(x)]_a^b = \int_a^b f(x)\,dx. \quad \text{[from (2.13)]}$$

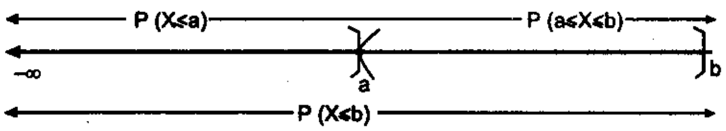


Fig. 2.5.

Moreover, it is evident that the area under the density function between a and b is just $P\{a \le X \le b\}$.

We note that the area under the curve is zero (when $a = b$) and hence $P\{a \le X \le b\} = P\{X = a\} = 0$.

Therefore, for continuous random variables, and any a and b,

$$P\{a \le X \le b\} = P\{a < X \le b\} = P\{a \le X < b\}$$
$$= P\{a < X < b\} = \int_a^b f(x)\,dx = F(b) - F(a).$$

But, this is not true in the case of discrete random variables.

Some immediate consequences for any random variable X with density function $f(x)$ are :

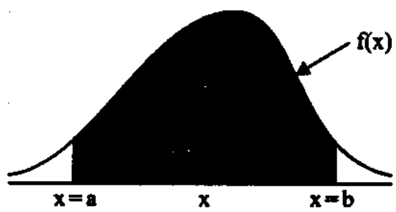


Fig. 2.6.

(*i*) $\int_{-\infty}^{\infty} f(x)\,dx = 1,$ ....(2.15)

(*ii*) $f(x)$ is non-negative, ...(2.16)

(*iii*) $P\{X = a\} = 0$, as pointed out earlier ....(2.17)

2.17 MEAN, VARIANCE AND STANDARD DEVIATION (S.D.)

If the random variable X assumes the discrete values $x_1, x_2, x_3, ..., x_r$ with corresponding probabilities $p_1, p_2, p_3, \ldots, p_r$ then,

$$\textbf{Mean (expected value)} = \bar{x} = \left[\sum_{i=1}^{r} p_i x_i \quad \sum_{i=1}^{r} p_i\right] = \sum_{i=1}^{r} p_i x_i \qquad (\because \Sigma p_i = 1) \qquad ...(2.18)$$

$$\textbf{Var}(x) = \sigma_x^2 = \sum_{i=1}^{r} (x_i - \bar{x})^2 p_i = \sum_{i=1}^{r} x_i^2 p_i - (\bar{x})^2 \textbf{ (when } \bar{x} \textbf{ is in fraction)} \qquad ...(2.19)$$

$$\textbf{S.D.} = \sigma_x = \sqrt{(\textbf{Var})}\,. \qquad ...(2.20)$$

In the case of continuous random variable X with probability density function $f(x)$, we have

$$\textbf{Mean} = \bar{x} = \int_{-\infty}^{\infty} x f(x)\,dx. \qquad ...(2.21)$$

$$\textbf{Var}(x) = \sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)\,dx \textit{ (definition)} \qquad ...(2.22a)$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - (\bar{x})^2 \textit{ (in computation)} \qquad ...(2.22b)$$

$$\textbf{S.D.} = \sigma_x = \sqrt{(\textbf{Var.})} \qquad ...(2.23)$$

EXAMINATION PROBLEMS

1. A win is so weighted that head is thrice as likely to appear as tail. What is *P*(*H*) and *P*(*T*).
   [Ans. ¾, ¼.]
2. There are 26 persons in a birthday party. What is the probability that at least two of them have the same birthday ?
   [Ans. 0.60]
3. If the probability that *A* will solve a problem is 1/4 and the probability that *B* will solve it is 3/4, what is the probability that the problem is at all solved ?
   [Ans. 13/16.]
4. An urn contains 3 green and 5 red balls. One ball is drawn, its colour unnoted and laid aside. Then another ball is drawn, find the probability that it is green or red. How does the probability change if the colour of the ball is noted ?
   [Ans. 3/8, 5/8, 2/7.]
5. An urn contains *a* white and *b* black balls. Balls are drawn one by one until only those of the same colour are left. What is the probability that they are white ?
   [Ans. $a/(a + b)$]
6. A pair of fair dice is rolled once. What is the probability that the sum is equal to each of the integers from 2 to 12 ?
   [Ans.

S :	2	3	4	5	6	7	8	9	10	11	12
p :	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

7. An urn contains one white and 2 black balls, while another contains 2 white and 1 black ball. One ball is transferred from the first urn into second, after which a ball is drawn from the second urn. What is the probability that it is black?
   [Ans. 5/12.]
8. Three players *A*, *B* and *C* play a sequence of games. It is also decided that winner of each game scores one point and he who first scores three points is the final winner. A wins the first and third games while *B* wins the second. What is the probability that *C* is the final winner ?
   [Ans. 2/27.]
9. A die is so loaded that the probability of a particular number appearing is proportional to the number. What is the probability of all single element events ? What is the probability of occurrence of an even number and of a number greater than 4 ?
   [Ans. 1/21, 2/21, 3/21, 4/21, 5/21, 6/21, 12/21, 11/21.]
10. A coin is tossed until a head appears, or until it has been tossed 3 times. If the head does not appear on the first toss, find the probability that the coin is tossed 3 times.
    [Ans. 1/2.]

11. Eight white and 2 black balls are randomly laid out in a row. What is the probability that two black balls are side by side ? What is the probability that they occupy the end positions ?
[Ans. 1/5, 1/45.]

12. An urn contains 10 white and 3 black balls white another contains 3 white and black balls. Two balls are transferred from the first urn to the second and then one ball is drawn from the latter, what is the probability that it is white ?
[Ans. 59/130.]

13. Find the value of c so that the following $f(x)$ is *p.d.f.*

$$f(x) = \begin{cases} c/x^2, & 10 \le x \le 20 \\ 0, & \text{otherwise.} \end{cases}$$

[Ans. $c = 20$.]

14. The following *p.d.f.* of the discrete random variable x represents the weekly demand of a certain item :

x	:	0	1	2	3
$P(x)$	:	0.15	0.25	0.35	0.45

If the weekly demands are independent and identical, find the *p.d.f.* for a two-week demand.

15. Eight coins were tossed together, and the number x of heads resulting was observed. The operation was performed 256 times, and the frequencies that were obtained for the different values of x are shown in the following table. Calculate measures of central tendency and mean deviation about mean–

x	:	0	1	2	3	4	5	6	7	8
f	:	1	9	26	59	72	52	29	7	1 (=256)

[Ans. $\bar{x} = 3.973$, Mean deviation (about mean) = 1.089] **[IAS (Maths.) 99]**

2.18 MATHEMATICAL EXPECTATION OF A RANDOM VARIABLE

For a *discrete random variable X*, the expected value is denoted by $E(X)$ which is just the sum of the products of the possible values the random variable X takes on and their respective associated probabilities.

In other words, if the discrete random variable X takes n mutually exclusive values $x_1, x_2, \ldots, x_n$, and no others, with respective probabilities $p_1, p_2, p_3, \ldots, p_n$, the expected value of x is given by

$$E(x) = p_1x_1 + p_2x_2 + p_3x_3 + \ldots + p_nx_n = \sum_{i=1}^{n} p_i x_i. \qquad \text{...(2.24)}$$

Similarly, for the continuous random variables, the expected value can also be obtained by the formula

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad \text{...(2.25)}$$

where $f(x)$ is the probability density function.

For example, if x has an exponential distribution with parameter a, the expected value is given by

$$E(x) = \int_0^{\infty} x\,(f(x)\,dx. = \int_0^{\infty} x\,(ae^{-ax})\,dx = \frac{1}{a}.$$

since the range for exponential distribution is 0 to ∞, and $f(x) = ae^{-ax}$.

Furthermore, we should also remember the following two expressions for the distribution function in *discrete* and *continuous* random variable cases :

(i) For a discrete random variable X, the *distribution* function denoted by $F(n)$, is given by

$$F(n) = P(X \le n) = \sum_{i=1}^{n} p_i. \qquad \text{...(2.26)}$$

(ii) For a continuous random variable X, the distribution function, denoted for $F(a)$ is given by

$$F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)\,dx. \qquad \text{...(2.27)}$$

Note. The mean of X is also called the **mathematical expectation** of X and is denoted by $E(X)$.

2.19 CENTRAL TENDENCY

After collecting the data for a random variable x and analysing it in the form of frequency distribution, the next necessary step is to find the nature of distribution. In this context, a property for values of x to tend towards the centre is quite important. This property is called the ***Central Tendency.*** There are three most important measures of central tendency : (*i*) mean, (*ii*) mode, (*iii*) median.

Mean. *If x is the random variable, then its expected value E(x) itself is called the **mean** or average value of x and denoted by $\bar{x}$. Mean value of the random variable locates the middle of its probability function.*

Mode. *The mode of a random variable x is that value of the variable which occurs with the greatest frequency and is denoted by $\hat{x}$. It is possible that a particular distribution may not have a mode, or if it has a mode, it may not be unique.*

A distribution is called unimodal, bimodal, trimodal, ... depending upon whether it has one, two, three .. ... modes.

For a discrete distribution, mode $\hat{x}$ is determined by the following inequalities :

(i) $P(x = x_i) \leq (x = \hat{x}), x_i \leq \hat{x}$, (ii) $P(x = x_j) \leq (x = \hat{x}), x_j \leq \hat{x}$.

For a continuous distribution it is determined by the following equations/inequalities :

$$\frac{d}{dx}[f(x)] = 0 \text{ and } \frac{d^2}{dx^2}[f(x)] = 0.$$

Median. *For a discrete or continuous distribution of a random variable x, the median is defined as the variate-value X satisfying the inequations : $P(x \leq X) = ½$ and $P(x \geq X) = ½$.*

It is denoted by $\tilde{x}$.

If a continuous distribution function has a *p.d.f. f(x)* in the range (*a, b*) then $\tilde{x}$ is given by

$$\int_a^x f(x)\,dx = ½ = \int_b^x f(x)\,dx.$$

Remark. If mean and median are known, then the mode can be calculated from the empirical formula

Mean – Mode = 3(Mean – Median).

SELF EXAMINATION PROBLEMS

1. What is the chance of throwing a total of 3 or 5 or 11 with two dice ?
2. *A, B, C* in order cut a pack of cards, replacing them after each cut. Find their respective chances of first cutting a heart.
3. Three cards are drawn from an ordinary pack. Find the chance that they consists of a knave, a queen and a king.
4. The odds against *A* solving a certain problem are 4 to 3 and odds in favour of *B* solving the same are 7 to 5. What is the chance that the problem will be solved if they both try ?
5. Four cards are drawn without replacement. What is the probability that (*a*) they are all aces ? (*b*) they are all of different suits ?
6. In shuffling a pack of cards, three are accidentally dropped. Find the chance that the missing cards should be from different suits.
7. A coin is tossed three times. Find the probability of getting head and tail alternately.
8. A bag contains 3 white, 4 red and 6 green balls. If only one ball is to be drawn, what is the chance of its being neither a red ball nor a green ball ?
9. An integer is chosen at random from 1 to 100. What is the probability that
   (*i*) it is divisible by 3, (*ii*) it is divisible by 8, (*iii*) it is divisible by 3 or 8 ?
10. Discuss and criticise the following :
    $P(A) = 2/3,\ P(B) = 1/4,\ P(C) = 1/6$
    where *A, B* and *C* are mutually exclusive events.
11. Show that in a single throw with two dice the chances of throwing more than 7 is equal to that of throwing less than 7 each being 5/12.
12. A speaks truth in 75% and *B* in 80% of the cases. In what percentages of cases are they likely to contradict each other in stating the same fact ?
    [**Ans.** 7/20]
13. What is the probability of obtaining 2 heads and two tails when 4 coins are thrown ?
    [**Ans.**3/8]
14. If the chance that a vessel arrives at a port is 9/10, find the chance that out of 5 vessels expected, 4 at least will arrive safely ?
    [**Ans.** 0·91854]
15. If on an average 9 ships out of 10 return safe to port, what is the chance that out of 5 ships expected at least 3 will arrive ?
    [**Ans.** 0·99144]

16. In four throws, with a pair of dice what is the chance of throwing doublets twice at least ?
[**Ans.** 19/144]

17. In tossing 10 coins, what is the probability of having exactly 5 heads ?
[**Ans.** 63/256]

18. Twelve per cent of a given lot of manufactured goods are defective. What is the probability that in a sample of five such goods exactly one will be defective?
[**Ans.** 0·3598]

19. Three per cent of a given lot of manufactured parts are defective. What is the probability that in a sample of four items non will be defective ?
[**Ans.** 0·8853]

20. A dice is thrown 120 times and getting '1' or '5' is considered a success. Find the mean and the variance of the number of successes.
[**Ans.** Mean = 40, Variance = 80/3]

21. Four coins are tossed. Find the mean and the variance of the number of heads obtained.
[**Ans.** Mean = 2, Variance = 1]

22. Two cards are drawn simultaneously (without replacement) from a well-shuffled pack of 52 cards. Find μ and σ for the number of aces.
[**Ans.** Mean = 34/221, Variance = 400/4873]

23. (*a*) If ten fair coins are tossed, what is the probability that there are (*i*) exactly 3 heads, (*ii*) not more than 3 heads.
(*b*) The probability that an evening college student will be graduate is 0.4. Determine the prob. that out of 5 students (*i*) none, (*ii*) one and (*iii*) at least one will be graduate.
[**Ans.** (a) (i) 15/128, (ii) 11/64; (b) (i) 0·07776, (iv) 0·2592, (iii) 0·9224]

24. Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 20% of such fuses are defective.
[**Ans.** 11817433 e^{-40}]

25. A perfect cubic die is thrown a large number of times in sets of 8. The occurrence of 5 or 6 is called a success. In what proportion of the sets would you expect 3 success ?
[**Ans.** 1792/6561]

26. In a certain factory turning out razor blades there is a small chance of 1/100 for any blade to be defective. The blades are supplied in packets of ten. In a consignment of 10,000 packets from the factory, how many packets are expected to have one defective blade ?
[**Ans.** 0·09135]

27. A square sheet of tin, 20 centimeters wide, that contains 10 rows and 10 columns of circular holes, each 1 centimeter in diameter, with centres evenly spaced at a distance 2 centimeters apart. What is the probability :
(*i*) that a particle of sand (considered as a point) blown against the tin sheet will fall upon one of the holes and thus pass through;
(*ii*) that a ball of diameter 0.5 cm. thrown upon the sheet will pass thought without hitting the tin sheet. **[I.A.S. (Maths.) 81]**
[**Ans.** (i) 11/56, (ii) 11/224]

28. A bag contains 10 balls, either black or white, but it is not known how many of each. A ball is drawn at random and is white. If a second ball is drawn at random (without the first ball being returned to the bag), what is the probability, it also will be white ?
[**Ans.** (a – 1)/9, where a denotes the no. of white balls.] **[I.A.S. (Maths.) 82]**

29. A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Determine the probability that it is (*i*) red, (*ii*) white, (*iii*) blue, (*iv*) not red, (*v*) red or white. **[I.A.S. (Maths.) 84]**
[**Ans.** (i) 6/15, (ii) 4/15, (iii) 5/15, (iv) 9/15, (v) 10/15]

30. Two cards are drawn successively from a pack without replacing the first. If the first card is a spade, find the probability that the second card is also a spade. Find also the probability that both cards are spades. **[I.A.S. (Maths.) 85]**
[**Ans.** 1/17]

31. A and B throw alternately with a pair of ordinary dice. A wins if he throws 6 before B throws 7, and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61. **[I.A.S. (Maths.) 87]**
[**Ans.** 1/17]

32. If n letters are randomly placed in correctly addressed envelopes, prove that the probability that exactly r letters are placed in correct envelopes is given by

$$\frac{1}{r!}\sum_{k=0}^{n-r}(-1)^k\frac{1}{k!}\,;\; r=1, 2, .., n.$$ **[I.A.S. (Maths.) 89]**

33. Show that the expectation of the sum of two stochastic variates is equal to the sum of their expectations. **[I.A.S. (Maths.) 88]**

34. A box contains a white and b balck balls. c ball are drawn. Find the expectation of the number of white balls drawn. **[I.A.S. (Maths.) 88]**

35. The contents of urns A, B, C are as follows :
Urn A : 1 white, 2 black and 3 red balls.
Urn B : 2 white, 1 black and 1 red balls.
Urn C : 4 white, 5 balck and 3 red balls.
One urn is chosen at random and two balls are drawn. They happen to be white and red. What are the probabilities of their having come from Urns A, B and C ? What is the probability that they came from urn B or C ?
[Ans. 33/118, 55/118, 30/118, 85/118] **[I.A.S. (Maths.) 94, 89]**

36. A bag contains a coin of value M and a number of other coins whose total value is m. A person draws one at a time till he draws the coin M. Find the value of his expectation.
[Ans. $(M+\frac{1}{2}m)$] **[I.A.S. (Maths.) 89]**

37. An urn contains 3 white, 5 black and 2 red balls. Two persons draw the ball tern by tern without replacement. The person who will draw the white ball first will be considered the winner. The game will be considered without decision if red ball is drawn.
Assuming that : A_1 = {the person who starts the game is the winner}, A_2 = {the second player is the winner}, and B = {the game is without decision}; find $P(A_1)$, $P(A_2)$ and $P(B)$.
[Ans. (10/17, 7/17, 2/17)] **[I.A.S. (Maths). 90]**

38. Balls are taken one by one out of an urn containing a white balls and b black balls. What is the expectation of the number of black balls preceding the first white ball ? **[I.A.S. (Maths.) 90]**

39. A box contains k varieties of objects, the number of objects of each variety being the same. Then objects are drawn one at a time and put back before the next drawn. Denoting by n the smallest number of drawings which produce objects of all varieties, find $E(n)$, the expectation of n. **[I.A.S. (Maths.) 90]**

40. Find the expectation of the number of failures preceding the first success in an infinite series of independent trials, with constant probability p of success in each trial.
[Ans. (1 – p)/p] **[I.A.S. (Maths) 91]**

41. A building contractor receives bricks from 3 different suppliers — 35% from supplier A, 45% from supplier B and the remaining from supplier C. 90% of bricks supplied by A, 80% of those supplied by B and 95% of those supplied by C are according to specifications. A brick drawn at random is not according to specification. What is the probability that it came from B ?
[Ans. 2/3] **[I.A.S. (Maths) 91]**

42. Each coefficient in the equation $Ax^2 + Bx + C = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots.
[Ans. 43/216] **[I.A.S. (Maths) 92]**

43. Prove that if A, B and C are random events in a sample space and if A, B, C are pairwise independent and A is independent of $B \cup C$, then A, B and C are mutually independent. **[I.A.S. (Maths) 92]**

44. An event A is known to be independent of the events B, $B \cup C$ and $B \cap C$. Show that it is also independent of C. **[I.A.S. (Maths) 93]**

45. If the probability that an individual suffers a bad reaction from injection of a given serum is 0·001, determine the probability that out of 2000 individuals : (i) exactly 3 will suffer a bad reaction (ii) more than 2 will suffer a bad reaction.
[Ans. (i) $4e^{-2}/3$, (ii) $1 - 4e^{-2}$] **[I.A.S. (Maths) 93]**

46. A and B play a game of dice. A wins if he throws 11 with 3 dice before B throws 7 with 2 dice. B wins if he throws 7 before A throws 11. A starts the game and they throw alternately. What are the odds against A winning the game ultimately ?
[Ans. 7 : 6] **[I.A.S. (Maths.) 95]**

47. If x be one of the first hundred netural numbers chosen at random, find the probability that

$$x+\frac{100}{x}>50.$$

[I.A.S. (Maths.) 96]

48. A printing machine can print n 'letters', say $\alpha_1, \alpha_2, ..., \alpha_n$. It is operated by electrical impulses, each letter being produced by a different impulse. Assume that there exists a constant probability p of printing the correct letter and also asuume independence. One of the n impulses chosen at random, was fed into the machine twice and both times the letter α_1 was printed. compute the probability that the impulse chosen was meant to print α_1. **[I.A.S. (Maths.) 99]**

49. If two independent variates, x and y have Poisson distributions with means m_1 and m_2, find the distribution of the sum $x+y$.
[Ans. $x + y \sim P(m_1 + m_2)$] **[I.A.S. (Maths.) 99]**

50. Two unbiased coins are tossed once (independently) and the number x of heads that turned up is noted. A number is selected at random from x, $x+1$ and $x+2$. It Y is the number selected, find the joint distribution of x and y. Also obtain the expectation of xy. **[I.A.S. (Maths.) 2000]**

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3

What Is Operations Research ?

3.1 INTRODUCTION : THE HISTORICAL DEVELOPMENT

In order to understand 'what Operations Research (OR)* is today,' we must know something of its history and evolution. The main origin of Operations Research was during the Second World–War. At that time, the military management in England called upon a team of scientists to study the strategic and tactical problems related to air and land defence of the country. Since they were having very limited military resources, it was necessary to decide upon the most effective utilization of them, *e.g.* the efficient ocean transport, effective bombing, etc.

During World-War II, the Military Commands of U.K. and U.S.A. engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations. Their mission was to formulate specific proposals and plans for aiding the Military Commands to arrive at the decisions on optimal utilization of scarce military resources and efforts, and also to implement the decisions effectively. The OR teams were not actually engaged in military operations and in fighting the war. But, they were only advisors and significantly instrumental in winning the war to the extent that the scientific and systematic approaches involved in OR provided a good intellectual support to the strategic initiatives of the military commands. Hence OR can be associated with ***"an art of winning the war without actually fighting it"***.

As the name implies, 'Operations Research' (sometimes abbreviated OR) was apparently invented because the team was dealing with *research* on (military) *operations*. The work of this team of scientists was named as *Operational Research* in England.

The encouraging results obtained by the British OR teams quickly motivated the United States military management to start with similar activities. Successful applications of the U.S. teams included the invention of new fight patterns, planning sea mining and effective utilization of electronic equipment. The work of OR team was given various names in the United States : ***Operational Analysis, Operations Evaluation, Operations Research, Systems Analysis, Systems Evaluation, Systems Research, Systems Analysis, Systems Evaluation, Systems Research, and Management Science.*** The name Operations Research was and is the most widely used so we shall also use it here.

Following the end of war, the success of military teams attracted the attention of *Industrial* managers who were seeking solutions to their complex executive-type problems. The most common problem was : what methods should be adopted so that the total cost is minimum or total profits maximum ? The first mathematical technique in this field (called the *Simplex Method* of linear programming) was developed in 1947 by American mathematician, **George B. Dantzig**. Since then, new techniques and applications have been developed through the efforts and cooperation of interested individuals in academic institutions and industry both.

Today, the impact of OR can be felt in many areas. A large number of management consultimg firms are currently engaged in OR activities. Apart from military and business applications, the OR activities include transportation system, libraries, hospitals, city planning, financial institutions, etc. Many of the Indian industries making use of OR activity are : ***Delhi Cloth Mills, Indian Railways, Indian Airlines, Defence Organizations, Hindustan Lever, Tata Iron & Steel Co., Fertilizer Corporation of India***, etc.

In business and other organizations, OR scientists and specialists always remain enagaged in the background. But, they help the top management officials and other line managers in doing their 'fighting' job better.

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* The short word **'OR'** for **'Operations Research'** should not be confused with the word **'or'** throughout the book.

While making use of the techniques of OR, a mathematical model of the problem is formulated. This model is actually a simplified representation of the problems in which only the most important features are considered for reasons of simplicity. Then, an *optimal* or *most favourable* solution is found. Since the model is an idealized in stead of exact representation of real problem, the optimal solution thus obtained may not prove to be the best solution to the actual problem. Although, extremely accurate but highly complex mathematical models can be developed, but they may not be easily solvable. So from both the cost-minimising and mathematical simplicity point of view, it seems beneficial to develop a less accurate but simpler model, and to find a sequence of solutions consisting of a series of increasingly better approximations to the actual course of action. Thus, the apparent weaknesses in the initial solution are used to suggest improvements in the model, its input-data, and the solution procedure. A new solution is thus obtained and the process is repeated until the further improvements in the succeeding solutions become so small that it does not seem economical to make further improvements.

If the model is carefully formulated and tested, the resulting solution should reach to be good approximation to the ideal course of action for the real problem. Although, we may not get the best answers, but definitely we are able to find *the bad answers where worse exist.* Thus operations research techniques are always able to save us from worse situations of practical life.

Q. 1. Comment the following statements : **[Rewa (Maths.) 93]**
(i) O.R. is the art of winning war without actually fighting it.
(ii) O.R. is the art of finding bad answers where worse exist.
2. What is O.R. ? **[Garhwal 97, 96; Meerut (IPM) 90]**
3. Enumerate six applications of Operations Research (O.R.) and describe one briefly. **[IGNOU 2001 (June)]**

3.2 THE NATURE AND MEANING OF 'OR' [IPM (PGDBA)* 82, 81; Meerut (Math.) 82]

'OR' has been defined so far in various ways and it is perhaps still too young to be defined in some authoritative way. So it is important and interesting to give below a few opinions about the definition of OR which have been changed according to the development of the subject.

1. OR is a scientific method of providing executive departments with a *quantitative basis* for *decision* regarding the operations under their control. ***—Morse and Kimbal (1946)***
2. OR is a scientific method of providing executive with an *analytical* and *objective* basis for decisions. ***—P.M.S. Blackett (1948)***
3. The term 'OR' has hitherto-fore been used to connate various attempts to study operations of war by scientific methods. From a more general point of view, OR can be considered to be an attempt to study those *operations of modern society which involved organizations of men or of men and machines.* ***—P.M. Morse (1948)***
4. OR is the application of *scientific methods, techniques* and *tools* to problems involving the *operations of systems* so as to provide these in control of the operations with *optimum solutions* to the problem. ***—Churchman, Acoff, Arnoff (1957)***
5. OR is the art of giving *bad answers* to problems to which otherwise *worse answers* are given. ***—T. L. Saaty (1958)***
6. OR is a management *activity* pursued in two complementary ways—one half by the free and bold *exercise of commonsense* untrammelled by any routine, and other half by the application of a repertoire of *well established precreated methods* and techniques. ***—Jagjit Singh* (1968)**
7. OR is the *attack* of modern methods on *complex problems* arising in the *direction and management* to large systems of men, machines, materials, and money in industry, business and defence. The distinctive approach is to developed a *scientific model* of the system, incorporating measurements of factors such as chance and risk with which to predict and *compare* the outcomes of alternative *decisions, strategies* or *controls.* The purpose is to help management to determine its policy and actions scientifically. ***—Operations Research Quarterly* (1971)**

* The symbol **Q.** will stand for '**EXAMINATION QUESTIONS**' throughout the book.

8. Operations Research is the art of winning war without actually fighting it.
9. OR is an applied decision theory. It uses any *scientific mathematical or logical means* to attempt to cope with the problems that confront the executive when he tries to achieve a through going rationality in dealing with his decision problems. ***—Miller and Starr.***
10. OR is a scientific approach to problem solving for executive management. ***—H.M. Wagner***
11. OR is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the scientific method of analysis. ***—C. Kittel***
12. OR is the systematic method oriented study of the basic structure, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision making. ***—E.L. Arnoff & M.J. Netzorg***
13. OR is the application of *scientific methods* to problems arising from operations involving *integrated* systems of *men, machines and materials.* It normally utilizes the knowledge and skill of an inter-disciplinary research team to provide the managers of such systems with optimum operating solutions. ***—Fabrycky and Torgersen***
14. OR is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man–machine systems and OR workers are actively engaged in applying this knowledge to practical problems in business, government, and society. ***—OR Society of America***
15. OR is the application of scientific method by inter-disciplinary teams to problems involving the controls of organized (man-machine) systems so as to provide solutions which *best serve the purpose of the organization as a whole.* ***—Ackoff & Sasieni, (1968)***
16. OR utilizes the planned approach (*updated scientific method*) and an *inter-disciplinary* team in order to represent complex functional relationships as mathematical models for purpose of providing a *quantitative basis* for decision making and *uncovering new problems* for quantitative analysis. ***—Thieanf and Klekamp*** **(1975)**

Comments on Definitions of OR :

From all above opinions, we arrive at the conclusion that whatever else 'OR' may be, it is certainly concerned with optimization problems. ***A decision, which taking into account all the present circumstances can be considered the best one, is called an optimal decision.*** **(Note)**

There are three main reasons for why most of the definitions of Operations Research are not satisfactory.

(i) First of all, Operations Research is not a science like any well-defined physical, biological, social phenomena. While chemists know about atoms and molecules and have theories about their interactions; and biologists know about living organisms and have theories about vital processes, *operations researchers* do not claim to know or have theories about operations. Operations Research is not a scientific research into the control of operations. It is essentially a collection of mathematical techniques and tools which in conjunction with a system approach are applied to solve practical decision problems of an *economic* or *engineering* nature. Thus it is very difficult to define Operations Research precisely.

(ii) Operations Research is inherently inter-disciplinary in nature with applications not only in military and business but also in medicine, engineering, physics and so on. Operations Research makes use of experience and expertise of people from different disciplines for developing new methods and procedures. Thus, inter-disciplinary approach is an important characteristic of Operations Research which is not included in most of its definitions. Hence most of the definitions are not satisfactory.

(iii) Most of the definitions of Operations Research have been offered at different times of development of 'OR' and hence are bound to emphasise its only one or the other aspect.

For example, 8th of the above definitions is only concerned with war alone. First definition confines 'OR' to be a scientific methodology applied for making operational decisions. It has no concern about the characteristics of different operational decisions and has not described how the scientific methods are applied in complicated situations. Many more definitions have been given by various authors but most of them fail to consider all basic characteristics of 'OR'. However, with further development of 'OR' perhaps more precise definitions should be forthcoming.

Q. 1. (a) Give any three definitions of Operations Research and explain them. [Meerut (IPM) 91; Meerut (O.R.) 90]
(b) Give reasons for : why most of the definitions of Operations Research are not satisfactory.
2. Discuss the three approaches of MIS development. [CA (May) 2000]
3. What are the pre-requisites of a computer based MIS ? [MCI 2000]

3.3 MANAGEMENT APPLICATIONS OF OPERATIONS RESEARCH

Some of the areas of management decision making, where the 'tools' and 'techniques' of O R are applied, can be outlined as follows :

1. **Finance–Budgeting and Investments**
   (i) Cash-flow analysis, long range capital requirements, dividend policies, investment portfolios.
   (ii) Credit policies, credit risks and delinquent account procedures.
   (iii) Claim and complaint procedures.
2. **Purchasing, Procurement and Exploration**
   (i) Rules for buying, supplies and stable or varying prices.
   (ii) Determination of quantities and timing of purchases.
   (iii) Bidding policies.
   (iv) Strategies for exploration and exploitation of raw material sources.
   (v) Replacement policies.
3. **Production Management**
   (i) ***Physical Distribution***
      (a) Location and size of warehouses, distribution centres and retail outlets.
      (b) Distribution policy.
   (ii) ***Facilities Planning***
      (a) Numbers and location of factories, warehouses, hospitals etc.
      (b) Loading and unloading facilities for railroads and trucks determining the transport schedule.
   (iii) ***Manufacturing***
      (a) Production scheduling and sequencing.
      (b) Stabilization of production and employment training, layoffs and optimum product mix.
   (iv) ***Maintenance and Project Scheduling***
      (a) Maintenance policies and preventive maintenance.
      (b) Maintenance crew sizes.
      (c) Project scheduling and allocation of resources.
4. **Marketing**
   (i) Product selection, timing, competitive actions.
   (ii) Number of salesman, frequency of calling on accounts per cent of time spent on prospects.
   (iii) Advertising media with respect to cost and time.
5. **Personnel Management**
   (i) Selection of suitable personnel on minimum salary.
   (ii) Mixes of age and skills.
   (iii) Recruitment policies and assignment of jobs.
6. **Research and Development**
   (i) Determination of the areas of concentration of research and development.
   (ii) Project selection.
   (iii) Determination of time cost trade-off and control of development projects.
   (iv) Reliability and alternative design.

From all above areas of applications, we may conclude that OR can be widely used in taking timely management decisions and also used as a corrective measure. The application of this tool involves certain data and not merely a personality of decision maker, and hence we can say : ***OR has replaced management by personality.***

Q. 1. "Operations Research replaces Management by personality." Discuss.
2. Explain applications of O.R. in Industry. [Garhwal 97; Karnataka 95]
3. Describe the various approaches used for development of MIS. [MCI 2000]

3.4 MODELLING IN OPERATIONS RESEARCH

Definition. *A model in the sense used in OR is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect.*

Since a model is an abstraction of reality, it thus appears to be less complete than reality itself. For a model to be complete, it must be a representative of those aspects of reality that are being investigated.

The main objective of a model is to provide means for analysing the behaviour of the system for the purpose of improving its performance. Or, if a system is not in existence, then a model defines the ideal structure of this future system indicating the functional relationships among its elements. The reliability of the solution obtained from a model depends on the validity of the model in representing the real systems. A model permits to 'examine the behaviour of a system without interfering with ongoing operations.

Models can be classified according to following characteristics :

1. Classification by Structure

(i) *Iconic models.* *Iconic models* represent the system as it is by scaling it up or down (*i.e.*, by enlarging or reducing the size). In other words, it is an image.

For example, a toy airplane is an iconic model of a real one. Other common examples of it are : photographs, drawings, maps etc. A model of an atom is scaled up so as to make it visible to the naked eye. In a globe, the diameter of the earth is scaled down, but the globe has approximately the same shape as the earth, and the relative sizes of continents, seas, etc., are approximately correct.

The iconic model is usually the simplest to conceive and the most specific and concrete. Its function is generally descriptive rather than explanatory. Accordingly, it cannot be easily used to determine or predict what effects many important changes on the actual system.

(ii) *Analogue models.* The models, in which one set of properties is used to represent another set of properties, are called *analogue models.* After the problem is solved, the solution is reinterpreted in terms of the original system.

For example, graphs are very simple analogues because distance is used to represent the properties such as : *time, number, per cent, age, weight,* and many other properties. Contour-lines on a map represent the rise and fall of the heights. In general, analogues are less specific, less concrete but easier to manipulate than are iconic models.

(iii) *Symbolic (Mathematical) models.* The *symbolic or mathematical model* is one which employs a set of mathematical symbols (*i.e., letters, numbers,* etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equations to describe the behaviour (or properties) of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model.

The symbolic model is usually the easiest to manipulate experimentally and it is most general and abstract. Its function is more often explanatory rather than descriptive.

2. Classification by Purpose

Models can also be classified by purpose of its utility. *The purpose of a model may be* descriptive, predictive or prescriptive.

(i) *Descriptive models.* A descriptive model simply describe some aspects of a situation based on observations, survey, questionnaire results or other available data. The result of an oppenion poll represents a descriptive model.

(ii) *Predictive models.* Such models can answer 'what if' type of questions, *i.e.* they can make predictions regarding certain events. For example, based on the survey results, television networks such models attempt to explain and predict the election results before all the votes are actually counted.

(iii) *Prescriptive models.* Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive (or normative) model because it prescribes what the managers ought to do.

3. Classification by Nature of Environment

These are mainly of two types :

(i) ***Deterministic models.*** Such models assume conditions of complete certainty and perfect knowledge. For example, linear programming, transportation and assignment models are deterministic type of models.

(ii) ***Probabilistic (or Stochastic) models.*** These types of models usually handle such situations in which the consequences or payoff of managerial actions cannot be predicted with certainty. However, it is possible to forecast a pattern of events, based on which managerial decisions can be made. For example, insurance companies are willing to insure against risk of fire, accidents, sickness and so on, because the pattern of events have been compiled in the form of probability distributions.

4. Classification by Behaviour

(i) ***Static models.*** These models do not consider the impact of changes that takes place during the planning horizon, *i.e.* they are independent of time. Also, in a static model only one decision is needed for the duration of a given time period.

(ii) ***Dynamic models.*** In these models, *time* is considered as one of the important variables and admit the impact of changes generated by time. Also, in dynamic models, not only one but a series of interdependent decisions is required during the planning horizon.

5. Classification by Method of Solution

(i) ***Analytical models.*** These models have a specific mathematical structure and thus can be solved by known analytical or mathematical techniques. For example, a *general linear programming* model as well as the specially structured *transportation* and *assignment models are analytical models.*

(ii) ***Simulation models.*** They also have a mathematical structure but *they cannot be solved* by purely using the 'tools' and 'techniques' of mathematics. A simulation model is essentially computer assisted experimentation on a mathematical structure of a real time structure in order to study the system under a variety of assumptions.

Simulation modelling has the advantage of being more flexible than mathematical modelling and hence can be used to represent complex systems which otherwise cannot be formulated mathematically. On the other hand, simulation has the disadvantage of not providing general solutions like those obtained from successful mathematical models.

6. Classification by Use of Digital Computers

The development of the digital computer has led to the introduction of the following types of modelling in OR.

(i) ***Analogue and Mathematical models combined.*** Sometimes analogue models are also expressed in terms of mathematical symbols. Such models may belong to both the types (ii) and (iii) in **classification 1** above.

For example, **simulation model** is of analogue type but mathematical formulae are also used in it. Managers very frequently use this model to '*simulate*' their decisions by summarizing the activities of industry in a scale-down period.

(ii) ***Function models.*** Such models are grouped on the basis of the function being performed.

For example, a function may serve to acquaint to scientist with such things as–tables, carrying data, a blue-print of layouts, a program representing a sequence of operations (like in computer programming).

(iii) ***Quantitative models.*** Such models are used to measure the observations.

For example, degree of temperature, yardstick, a unit of measurement of length value, etc.

Other examples of quantitative models are : (i) *transformation models* which are useful in converting a measurement of one scale to another (*e.g.,* Centigrade *vs* Fahrenheit conversion scale), and (ii) the *test models* that act as 'standards' against which measurements are compared (*e.g.,* business dealings, a specified standard production control, the quality of a medicine).

(iv) ***Heuristic models.*** These models are mainly used to explore alternative strategies (courses of action) that were overlooked previously, whereas mathematical models are used to represent systems possessing well-defined strategies. Heuristic models do not claim to find the best solution to the problem.

Q. 1. Model building is the essence of the 'O.R. approach'. Discuss.

2. Discuss in detail the three types of models with special emphasis on the important logical properties and the relationship the three types bear to each other and to modelled entities. **[Meerut (OR) 90]**

3. What is meant by a mathematical model of real situation ? Discuss the importance of models in the solution of Operational Research probems ?

4. What is a model ? Discuss various classification schemes of models. **[Agra 95, 94; C.A. (May) 92; Meerut (IPM) 90]**

3.5 PRINCIPLES OF MODELLING

Let us now outline general principles useful in guiding to formulate the models within the context of O R. The model building and their users both should be consiously aware of the following ***Ten*** principles :

1. ***Do not build up a complicated model when simple one will suffice.*** Building the strongest possible model is a common guiding principle for mathematicians who are attempting to extend the theory or to develop techniques that have wide applications. However, in the actual practice of building models for specific purposes, the best advice is to "keep it simple".
2. ***Beware of molding the problem to fit the technique.*** For example, an expert on linear programming techniques may tend to view every problem he encounters as required in a linear programming solutions. In fact, not all optimization problems involve only linear functions. Also, not all OR problems involve optimization. As a matter of fact, not all real-world problems call for *operations research* ! Of course, every one search reality in his own terms, so the field of OR is not unique in this regard. Being human, we rely on the methods we are most comfortable in using and have been most successful within the past. We are certainly not able to use techniques in which we have no competence, and we cannot hope to be competent in all techniques. We must divide OR experts into three main categories :
   (i) Technique developers, (ii) Teachers, and (iii) Problem solvers.
   In particular, one should be ready to tolerate the behaviour "I have found a cure but I am trying to search a disease to fit it" among *technique developers* and *teachers.*
3. ***The deduction phase of modelling must be conducted rigorously.*** The reason for requiring rigorous deduction is that one wants to be sure that if model conclusions are inconsistent with reality, then the defect lies in the assumptions. One application of this principle is that one must be extremely careful when programming computers. Hidden "bugs" are specially dangerous when they do not prevent the program from running but simply produce results which are not consistent with the intention of the model.
4. ***Models should be validated prior to implimentation.*** For example, if a model is constructed to forecast the monthly sales of a particular commodity, it could be tested using historical data to compare the forecasts it would have produced to the actual sales. In case, if the model cannot be validated prior to its implementation, then it can be implemented in phases for validation. For example, a new model for inventory control may be implemented for a certain selected group of items while the older system is retained for the majority of remaining items. If the model proves successful, more items can be placed within its range. It is also worthnoting that real things change in time. A highly satisfactory model may very well degrade with age. So periodic re-evaluation is necessary.
5. ***A model should never be taken too literally.*** For example, suppose that one has to construct an elaborate computer model of Indian economy with many competent researchers spending a great deal of time and money in getting all kinds of complicated interactions and relationships. Under such circumstances, it can be easily believed as if the model duplicates itself the real system. This danger continues to increase as the models become larger and more sophisticated, as they must deal with increasingly complicated problems.
6. ***A model should neither be pressed to do, nor criticized for failing to do that for which it was never intended.*** One example of this error would be the use of forecasting model to predict so far into the future that the data on which the forecasts are based have no relevance. Another example is the use of certain network methods to describe the activities involved in a complex project. A model should not be stretched beyond its capabilities.
7. ***Beware of over-selling a model.*** This principle is of particular importance for the OR professional because most non-technical benefactors of an operations researcher's work are not likely to understand his methods. The increased technicality of one's methods also increases the burden of responsibility on the OR. professional to distinguish clearly between his role as model manipulator and model interpreter. In those cases where the assumptions can be challenged, it would be dishonest to use the model.
8. ***Some of the primary benefits of modelling are associated with the process of developing the model.*** It is true in general that a model is never as useful to anyone else as it is to those who are involved in building it up. The model itself never contains the full knowledge and understanding of the real system that the builder must acquire in order to successfully model it, and there is no practical way to convey this knowledge and understanding properly. In some cases, the sole benefits may occur while the model is

being developed. In such cases, the model may have no further value once it is completed. An example of this case might occur when a small group of people attempts to develop a formal plan for some subject. The plan is the final model, but the real problem may be to agree on 'what the objectives ought to be'.

9. ***A model cannot be any better than the Information that goes into it.*** Like a computer program, a model can only manipulate the data provided to it; it cannot recognize and correct for deficiencies in input. Models may *condense* data or *convert* it to more useful forms, but they do not have the capacity to generate it. In some situations it is always better to gather more information about the system instead of exerting more efforts on modern constructions.
10. ***Models cannot replace decision makers.*** The purpose of OR models should not be supposed to provide "Optimal solutions" free from human subjectivity and error. OR models can aid decision makers and thereby permit better decisions to be made. However, they do not make the job of decision making easier. Definitely, the role of experience, intuition and judgement in decision making is undiminished.

3.6 APPROXIMATIONS (SIMPLIFICATIONS) OF 'OR' MODLES

While constructing a model, two conflicting objectives usually strike in our mind :

(i) The model should be as accurate as possible.

(ii) It should be as easy as possible in solving.

Besides, the management must be able to understand the solution of the model and must be capable of using it. So the reality of the problem under study should be simplified to the extent when there is no loss of accuracy.

The model can be simplified by :

(i) omitting certain variable *(ii)* changing the nature of variables *(iii)* aggregating the variables *(iv)* changing the relationship between variables, and *(v)* modifying the constraints, etc.

3.7 GENERAL METHODS FOR SOLVING 'OR' MODLES

Generally, three types of methods are used for solving OR models.

Analytic Method. If the OR model is solved by using all the tools of classical mathematics such as : differential calculus and finite differences available for this task, then such type of solutions are called *analytic solutions.* Solutions of various inventory models are obtained by adopting the so called analytic procedure.

Iterative Method. If classical methods fail because of complexity of the constraints or of the number of variables, then we are usually forced to adopt an iterative method. Such a procedure starts with a trial solution and a set of rules for improving it. The trial solution is then replaced by the improved solution, and the process is repeated until either no further improvement is possible or the cost of further calculation cannot be justified.

Iterative method can be divided into three groups :

(a) After a finite number of repetitions, no further improvement will be possible.

(b) Although successive iterations improve the solutions, we are only guaranteed the solution as a limit of an infinite process.

(c) Finally, we include the *trial* and *error* method which, however, is likely to be lengthy, tedious, and costly even if electronic computers are used.

The Monte-Carlo Method. The basis of so called Monte-Carlo technique is random sampling of variable's values from a distribution of that variable. Monte-Carlo refers to the use of sampling methods to estimate the value of non-stochastic variables. The following are the main steps of Monte-Carlo method :

Step 1. In order to have a general idea of the system, we first draw a *flow diagram* of the system.

Step 2. Then, we take correct sample observations to select some suitable model for the system. In this step, we compute the probability distributions for the variables of our interest.

Step 3. We, then, convert the probability distributions to a cumulative distribution function.

Step 4. A sequence of random numbers is now selected with the help of random number tables.

Step 5. Next, we determine the sequence of values of variables of interest with the sequence of random numbers obtained in **step 4.**

Step 6. Finally, we construct some standard mathematical function to the values obtained in ***step 5.***

Q. 1. State the different types of models used in OR. Explain briefly the general methods for solving these O.R. models. **[Agra 95]**

2. Write briefly about the following :
(i) Iconic models (ii) Analogue models (iii) Mathematical models **[Meerut (MCA III) May 2000]**

The following interesting example will make the above procedure clear.

Illustration of Monte-Carlo Technique

Example. *A bombing mission is sent to bomb an important factory, which is rectangular in shape and has the dimensions 250 by 500 feet. The bombers will drop 10 bombs altogether, from high altitude, all aimed at the geometric centre of the plant. We assume that the bombing run is made parallel to the long dimension of the plant, that the deviation of the impact point from the aiming point is normal with mean zero and standard deviation 200 feet in each dimension, and that these two deviations are independent random variables. Use Monte-Carlo sampling to estimate the expected number of bomb-hits, and compare your result with the exact value.*

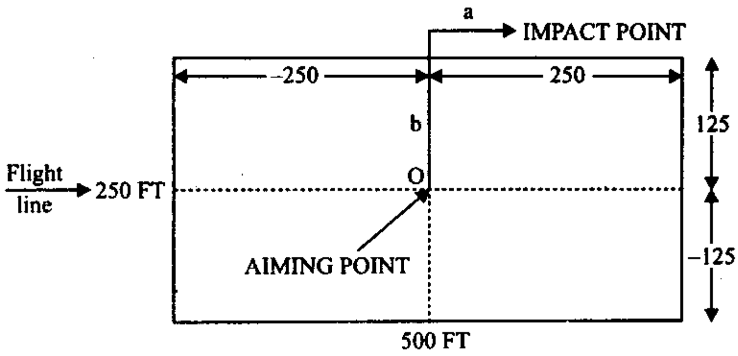


Fig. 3.1.

Solution. Let a be the horizontal deviation, and b be the vertical deviation, as shown in the Fig. 3.1.

The bomb will strike if the following conditions are satisfied :

$$-250 \le a \le 250, \quad -125 \le b \le 125, \quad ...(3\cdot1)$$

otherwise the bomb will miss the target.

If we put $x = a/200$, $y = b/200$, so that x and y will be the corresponding deviates read from a random number table, then the condition (3·1) for hitting the target becomes :

$$-250 \le 200x \le 250, \quad -125 \le 200y \le 125 \quad ...(3\cdot2a)$$

or

$$-1\cdot250 \le x \le 1\cdot250, \quad -0\cdot625 \le y \le 0\cdot625 \quad ...(3\cdot2b)$$

Results for the first three trials are given in Table 3·1 below.

Table 3·1

Bomb	*x*	*y*	*Result*
Trial 1, four hits			
1	– 0·291	1·221	Miss
2	– 2·828	– 0·439	Miss
3	0·247	1·291	Miss
4	– 0·584	0·541	Hit*
5	– 0·446	– 2·661	Miss
6	– 2·127	0·665	Miss
7	0·656	0·340	Hit*
8	1·041	0·008	Hit*
9	0·899	0·110	Hit*
10	– 1·114	1·297	Miss
Trial 2, two hits			
1	1·119	0·004	Hit*
2	– 0·792	– 1·275	Miss
3	0·063	– 1·793	Miss
4	0·484	– 0·986	Miss
5	1·045	– 2·363	Miss
6	– 0·084	– 0·880	Miss
7	– 0·086	– 0·158	Hit*
8	0·427	– 0·831	Miss
9	– 0·528	– 0·833	Miss
10	– 1·433	– 1·345	Miss

Trial 3, four hits	x	y	*Result*
1	– 2·015	– 0·594	Miss
2	– 0·623	– 1·047	Miss
3	– 0·699	– 1·347	Miss
4	0·481	0·996	Miss
5	0·586	– 1·023	Miss
6	0·579	0·551	**Hit***
7	0·120	0·418	**Hit***
8	0·191	0·074	**Hit***
9	0·071	0·524	**Hit***
10	– 3·001	0·479	Miss

These three trials give 3·33 as the average number of hits per mission. Many more trials should be conducted before we can have any real confidence in the result. One way of estimating : how many trials are necessary, is to list the cumulated mean at the end of each trial, and to stop the trials when the mean seems to have settled down to stable value. In this example, we have

After trial number	:	1	2	3
Cummulated mean	:	4	3	3·33,

so that more trials are necessary.

The mean number of hits in a mission dropping 10 bombs is 3·69.

To compare the result with the exact value.

In this problem, unlike most Monte-Carlo problems, an exact calculation of the answer is much easier than the Monte Carlo calculation.

The probability of a hit with a single bomb is

$$\left[\int_{-1.250}^{1.250} f(x)\,dx\right] \times \left[\int_{-0.625}^{0.625} f(x)\,dx\right] = 2.789 \times 0.468 \qquad \text{(from the table of the normal integers)}$$

$$= 0.369.$$

Thus the previous value 3·69 is ten times of this value.

Advantages :

1. These methods avoid unnecessary expenses and difficulties that arise during the trial and error experimentation.
2. By this technique, we find the solution of much complicated mathematical expression which is not possible by any other method.

Disadvantages :

1. This technique does not give optimal answers to the problems. The good results are obtained only when the sample size is quite large.
2. The computations are much complicated even in simple cases.
3. It is a costly procedure for obtaining a solution of any related problem.

Q. 1. Write a short note on Monte-Carlo Technique and their usefulness in real life situations. **[Meerut (Stat.) 98]**

2. Describe the use of Monte-Carlo methods in sampling experiments. Illustrate with possible examples.

3.8 MAIN CHARACTERISTICS (FEATURES) OF OPERATIONS RESEARCH

The main characteristics of OR are as follows :

1. Inter-disciplinary team approach. In **OR**, the optimum solution is found by a team of scientists selected from various disciplines such as mathematics, statistics, economics, engineering, physics, etc.

For example, while investigating the inventory management in a factory, perhaps we may require an engineer who knows the functions of various items of stores. We also require a cost accountant and a mathematician-cum-statistician. Each member of such OR team is benefitted by the view points of others, so that the workable solution obtained through such collaborative study has a greater chance of acceptance by management.

Furthemore, an OR team required for a big organization may include a statistician, an economist, a mathematician, one or more engineers, a psychologist, and some supporting staff like computer programmers, etc. A mathematician or a probabilist can apply his tools in a plant problem only if he gets to understand some of the physical implications of the plant from an engineer. Otherwise, he may give such a solution which may not be possible to apply.

2. Wholistic approach to the system. The most of the problems tackled by OR have the characteristic that OR tries to find the *best* (*optimum*) decisions relative to largest possible portion of the total organization. The nature of organization is essentially immaterial.

For example, in attempting to solve a maintenance problem in a factory, OR tries to consider how this affects the production department as a whole. If possible, it also tries to consider how this effect on the production department in turn affects other department and the business as a whole. It may even try to go further and investigate how the effect on this particular business organization in turn affects the industry as a whole, etc. Thus OR attempts to consider inter-actions or chain of effects as far out as these effects are significant.

3. Imperfectness of solutions. By OR techniques, we cannot obtain perfect answers to our problems but, only the quality of the solution is improved from worse to bad answers.

4. Use of scientific research. OR uses techniques of scientific research to reach the optimum solution.

5. To optimize the total output. OR tries to optimize total return by maximizing the profit and minimizing the cost or loss.

Q. 1. Give the main characteristics of Operations Research. **[C.A. (May) 92]**

2. Define OR and discuss its characteristics and limitations.

3.9 MAIN PHASES OF OPERATIONS RESEARCH STUDY

About fourty years ago, it would have been difficult to get a single operations-researcher to describe a procedure for conducting OR project. The procedure for an OR study generally involves the following major phases :

Phase I : Formulating the problem. Before proceeding to find the solution of a problem, first of all one must be able to formulate the problem in the form of an appropriate model. To do so, the following information will be required.

(i) Who has to take the *decision* ?

(ii) What are the *objectives* ?

(iii) What are the ranges of *controlled variables* ?

(iv) What are the uncontrolled variables that may affect the possible solutions ?

(v) What are the restrictions or constraints on the variables ?

Since wrong formulation cannot yield a right decision (solution), one must be considerably careful while execution this phase.

Phase II : Constructing a mathematical model. The second phase of the investigations is concerned with the reformulation of the problem in an appropriate form which is convenient for analysis. The most suitable form for this purpose is to construct a mathematical model representing the system under study. It requires the identification of both *static* and *dynamic* structural elements. A mathematical model should include the following three important basic factors :

(i) *Decision variables and parameters*, (ii) *Constraints or Restrictions*, (iii) *Objective function.*

Phase III : Deriving the solutions from the model. This phase is devoted to the computation of those values of decision variables that maximize (or minimize) the objective function. Such solution is called an *optimal solution* which is always in the best interest of the problem under consideration. The general techniques for deriving the solution of OR model are discussed in the following sections and further details are given in the text.

Phase IV : Testing the model and its solution (updating the model). After completing the model, it is once again tested as a whole for the errors if any. A model may be said to be valid if it can provide a reliable prediction of the system's performance. A good practitioner of Operations Research realises that his model be applicable for a longer time and thus he updates the model time to time by taking into account the past, present and future specifications of the problem.

Phase V : Controlling the solution. This phase establishes controls over the solution with any degree of satisfaction. The model requires immediate modification as soon as the controlled variables (one or more) change significantly, otherwise the model goes out of control. As the conditions are constantly changing in the world, the model and the solution may not remain valid for a long time.

Phase VI : Implementing the solution. Finally, the tested results of the model are implemented to work. This phase is primarily executed with the cooperation of Operations Research experts and those who are responsible for managing and operating the systems.

Q. 1. Discuss the various phases in solving an OR problem. [IGNOU 2001; C.A. (Nov.) 92; Meerut (IPM) 90]
2. What are various phases of O.R. problems ? Explain them briefly. [VTU (BE Mech.) 2003]
3. Give the different phases of Operations Research, and explain their significance in decision making. [Meerut (Stat.) 98, 90; Karnataka (B.E.) 95; C.A. (Nov.) 89]
4. Explain the steps involved in the solution of an Operations Research problem. [IGNOU 2001]
5. What is an operations research? Discuss the various phases in solving an OR problem. [AIMS (B.E.) Bangalore 2002]

3.10 THE TERMS : 'TOOLS', 'TECHNIQUES' AND 'METHODS'

We now carefully differentiate the terms : 'tools', 'techniques' and 'methods' which are frequently used in science. It is evident that a table of random numbers is a *tool* of science. The way in which this tool is used is called a *technique*. The research plan which involves the use of ***Monte-Carlo*** procedure and the table of random numbers is called a *method* of science. Similarly, calculus is a scientific *tool*; employing calculus to find an optimum value of a variable in a mathematical model of a system is a scientific *technique*; and the plan of utilizing a mathematical model to optimize a system is a scientific *method.*

3·10-1 Scientific Method in Operations Research

The scientific method in OR study generally involves the three phases : (i) *the judgement phase*, (ii) *the research phase*, and (iii) *the action phase.*

Of these three, the *research phase* is the largest and longest, but the remaining two are just as important as they provide the basis for an implementation of the research.

The judgment phase includes :

(i) A determination of the operation.
(ii) The establishment of the objectives and values related to the operation.
(iii) The determination of the suitable measures of effectiveness.
(iv) Lastly, the formulation of the problems relative to the objectives.

The research phase utilizes :

(i) Observations and data collection for a better understanding of what the problem is.
(ii) Formulation of hypothesis and models.
(iii) Observation and experimentation to test the hypothesis on the basis of additional data.
(iv) Analysis of the available information and verification of the hypothesis using pre-established measures of effectiveness.
(v) Predictions of various results from the hypothesis, generalization of the result and consideration of alternative methods.

The action phase :

OR consists of making recommendations for decision process by those who first posed the problem for consideration, or by anyone in a position to make a decision influencing the operation in which the problem occured.

Q. 1. Discuss Scientific Method in O R. [Meerut (O.R.) 90]
2. Enumerate the approach, technique and tools used in operations research. You may list as many as possible but focus on 4 tools and detail the appropriate computer hardware, software and application programs. [IGNOU 2001, 99, 96]

3.11 SCOPE OF OPERATIONS RESEARCH

In its recent years of organized development, OR has entered successfully many different areas of research for military, government and industry. The basic problem in most of the developing countries in Asia and Africa is to remove *poverty* and *hunger* as quickly as possible. So there is a great scope for economists, statisticians, administrators, politicians and the technicians working in a team to solve this problem by an OR approach. Besides this, OR is useful in the following various important fields.

1. In Agriculture. With the explosion of population and consequent shortage of food, every country is facing the problem of—

(i) optimum allocation of land to various crops in accordance with the climatic conditions; and
(ii) optimum distribution of water from various resources like canal for irrigation purposes.

Thus there is a need of determining best policies under the prescribed restrictions. Hence a good amount of work can be done in this direction.

2. In Finance. In these modern times of economic crisis, it has become very necessary for every government to have a careful planning for the economic development of her country. OR-techniques can be fruitfully applied :

(i) to maximize the per capita income with minimum resources;
(ii) to find out the profit plan for the company;
(iii) to determine the best replacement policies, etc.

3. In Industry. If the industry manager decides his policies (not necessarily optimum) only on the basis of his past experience (without using OR techniques) and a day comes when he gets retirement, then a heavy loss is encountered before the Industry. This heavy loss can immediately be compensated by newly appointing a young specialist of OR techniques in *business management.* Thus OR is useful to the *Industry Director* in deciding optimum allocation of various limited resources such as men, machines, material, money, time, etc., to arrive at the optimum decision.

4. In Marketing. With the help of OR techniques a *Marketing Administrator* (Manager) can decide :
(i) where to distribute the products for sale so that the total cost of transportation etc. is minimum,
(ii) the minimum per unit sale price,
(iii) the size of the stock to meet the future demand,
(iv) how to select the best advertizing media with respect to time, cost, etc.
(v) how, when, and what to purchase at the minimum possible cost ?

5. In Personnel Management. A personnel manager can use OR techniques :
(i) to appoint the most suitable persons on minimum salary,
(ii) to determine the best age of retirement for the employees,
(iii) to find out the number of persons to be appointed on full time basis when the workload is seasonal (not continuous).

6. In Production Management. A production manager can use OR techniques :
(i) to find out the number and size of the items to be produced;
(ii) in scheduling and sequencing the production run by proper allocation of machines;
(iii) in calculating the optimum product mix; and
(iv) to select, locate, and design the sites for the production plants.

7. In L.I.C. OR approach is also applicable to enable the L.I.C. offices to decide :
(i) what should be the premium rates for various modes of policies,
(ii) how best the profits could be distributed in the cases of with profit policies ? etc.

Finally, we can say : wherever there is a problem, there is OR. The applications of OR cover the whole extent of any thing. A recent publication of the OR society contains a summary of the applications of OR. The reader wishing more details on applications may consult the publication : ***'Progress in OR'* Vol. 2 by *Hertz., D.B.* and *R.T. Eddison.***

Q. 1. Define O.R. and discuss its scope. **[Meerut (Stat.) 98; Garhwal 96; Kanpur 96; Rewa (Maths.) 93; Rohit. 93, 92]**
2. What are the areas of applications of O.R., **[Meerut (Maths) 91]**
3. (a) Explain the meaning, scope and methodology of O.R. **[VTU (BE Mech.) 2002]**
(b) Discuss the significance and scope of Operations Research in modern management.
4. Write a critical essay on the definition and scope of Operations Research. **[JNTU (B. Tech) 2002; Virbhadrah 2000]**

3.12 ROLE OF OPERATIONS RESEARCH IN DECISION-MAKING

The Operations Research may be regardeed as, a tool which is utilized to increase the effectiveness of management decisions. In fact, OR is the objective suppliment to the subjective feeling of the administrator (decision-maker). Scientific method of OR is used to understand and describe the phenomena of operating system. OR models explain these phenomena as to what changes take place under altered conditions, and control these predictions against new observations. For example, OR may suggest the best locations for factories, warehouses as well as the most economical means of transportation. In marketing, OR may help in indicating the most profitable type, use and size of advertising compaigns subject to the financial limitations.

The advantages of OR study approach in business and management decision making may be classified as follows :

1. Better Control. The management of big concerns finds it much costly to provide continuous executive supervisions over routine decisions. An OR approach directs the executives to devote their attention to more pressing matters. For example, OR approach deals with production scheduling and inventory control.

2. Better Co-ordination. Sometimes OR has been very useful in maintaining the law and order situation out of chaos. For example, an OR based planning model becomes a vehicle for coordinating marketing decisions with the limitations imposed on manufacturing capabilities.

3. Better System. OR study is also initiated to analyse a particular problem of decision making such as establishing a new warehouse. Later, OR approach can be further developed into a system to be employed repeatedly. Consequently, the cost of undertaking the first application may improve the profits.

4. Better Decisions. OR models frequently yield actions that do improve an intuitive decision making. Sometimes, a situation may be so complicated that the human mind can never hope to assimilate all the important factors without the help of OR and computer analysis.

In the present text, we restrict ourselves to discuss the problems on : ***Inventory control, Replacement, Queues, Linear programming, Goal Programming, Transportation, Assignment, Games theory, Sequencing, Dynamic programming, Information theory, PERT/CPM, Simulation, and Decision theory*** etc.

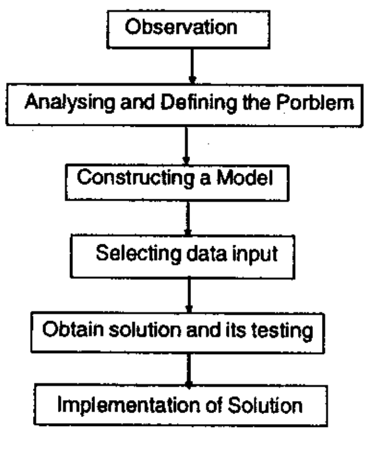
O.R. provides a logical and systematic approach for decision-making. The phases and process study must also be quite logical and systematic. There are six important steps in O.R. study, but in all each and every step does not necessarily follow logical order as below :

***Step I* : Observing the Problem Environment**

The activities in this step are visits, conferences, observations, research etc. With such activities analyst gets sufficient information and support to define the problem.

***Step II* : Analysing and Defining the Problem**

In this step the problem is defined, and objectives and limitations of the study are stated in its context. One thus gets clear grasp of need for a solution and indication of its nature.



***Step III* : Developing a Model**

Step III is to contruct a model. A model is representation of some real or abstract situations. O.R. models are basically mathematical models representing systems, processes or environment in the form of equations, relationships or formulae. The activities in this step are defining interrelationships among variables, formulating equations, using known O.R. models or searching suitable alternate models. The proposed model may be practically tested and modified in order to work under given environmental constraints. A model may also be managerial if not satisfied with the solution it offers.

***Step IV* : Selecting Appropriate Data Input**

No model will work appropriately if data input is not appropriate. Hence using right kind of data is vital in O.R. process, Important activities in this step are analysing internal-external data and facts, collecting openions and using computer data banks. The purpose of the step is to have sufficient input to operate and test the models.

***Step V* : Providing a Solution and Testing its Reasonableness**

Step V is performed to obtain a solution with the help of model and data input. Such a solution is not implemented immediately. First it is tested not behaving properly, updating and modification of the model is considered at this stage. The end result is solution that supports current organization objective.

***Step VI* : Implementing the Solution**

Implementation of the solution is the last step of O.R. process. In O.R., the decision-making is scientific, but implementation of decision involves many behavioural issues. Therefore, the implementing authority has to resolve the behavioural issues. He has to convince not only the workers but also the superiors. The gap between one who provides a solution and the other who wishes to use it has to be eliminated. To achieve this, O.R. analyst as well as management should play a positive role. Needless to say a properly implemented solution obtained through O.R. techniques results an improved working and gets active management support.

Q. 1. What is the importance (role) of Operations Research in decision making. **[JNTU (BE Comp. Sc.) 2004; Kanpur 96]**

2. Describe in brief the role of quantitative techniques in bussiness management.
3. What are the various phases through which an O.R. team normally has to proceed ?
4. Give any four examples of management problems solved through OR. **[JNTU (BE Comp. Sc.) 2004]**

3.13 BRIEF OUTLINES OF OR-MODELS : QUANTITATIVE TECHNIQUES OF OR

A brief account of some of the important OR models is given below :

1. Distribution (Allocation) Models. Distribution models are concerned with the allotment of available resources so as to minimise cost or maximise profit subject to prescribed restrictions. Methods for solving such type of problems are known as *mathematical programming techniques*. We distinguish between linear and non-linear programming problems on the basis of linearity and non-linearity of the objective function and/or constraints respectively. In linear programming problems, the objective function is linear and constraints are also linear inequalities/equations . Transportation and Assignment models can be viewed as special cases of linear programming. These can be solved by specially devised procedures called *Transportation* and *Assignment Techniques.*

In case the decision variables in a linear programming problem are restricted to either integer or zero-one value, it is known as *Integer* and *Zero-One programming problems*, respectively. The problems having multiple, conflicting and incommensurable objective functions (goals) subject to linear constraints are called *linear goal programming problems*. If the decision variables in a linear programming problem depend on chance, then such problems are called *stochastic linear programming problems.*

2. Production/Inventory Models. Inventory/Production models are concerned with the determination of the optimal (economic) order quantity and ordering (production) intervals considering the factors such as–demand per unit time, cost of placing orders, costs associated with goods held up in the inventory and the cost due to shortage of goods, etc. Such models are also useful in dealing with quantity discounts and multiple products.

3. Waiting Line (or Queueing) Models. In queueing models an attempt is made to predict :

(i) how much average time will be spent by the customer in a queue ?
(ii) what will be an average length of waiting line or queue ?
(iii) what will be the traffic intensity of a queueing system ? etc.

The study of waiting line problems provides us methods to minimize the sum of costs of providing service and cost of obtaining service which are primarily associated with the value of time spent by the customer in a queue.

4. Markovian Models. These models are applicable in such situations where the state of the system can be defined by some descriptive measure of numerical value and where the system moves from one state to another on a probability basis. Brand-swiching problems considered in marketing studies is an example of such models.

5. Competetive Strategy Models (Games Theory). These models are used to determine the behaviour of decision-making under competition or conflict. Methods for solving such models have not been found suitable for industrial applications, mainly because they are referred to an idealistic world neglecting many essential features of reality.

6. Network Models. These models are applicable in large projects involving complexities and inter-dependencies of activities. *Project Evaluation and Review Techniques* (PERT) and *Critical Path Method* (CPM) are used for planning, scheduling and controlling complex project which can be characterised as net-works.

7. Job Sequencing Models. These models involve the selection of such a sequence of performing a series of jobs to be done on service facilities (machines) that optimize the efficiency measure of performance of the system. In other words, sequencing is cencerned with such a problem in which efficiency measure depends upon the order or sequence of performing a series of jobs.

8. Replacement Models. These models deal with the determination of optimum replacement policy in situations that arise when some items or machinery need replacement by a new one. Individual and group replacement policies can be used in the case of such equipments that fail completely and instantaneously.

9. Simulation Models. Simulation is a very powerful technique for solving much complex models which cannot be solved otherwise and thus it is being extensively applied to solve a variety of problems. This technique is more useful when following two types of difficulties may arise :

(i) The number of variables and constraint relationships may be so large that it is not computationally feasible to pursue such analysis.

(ii) Secondly, the model may be much away from the reality that no confidence can be placed on the computational results.

In fact, such models are solved by simulation techniques where no other method is available for its solution.

Operations Research, as its name suggests, gives stress on analysis of operations as a whole. For this purpose it uses any suitable techniques or tools available from the fields of mathematics, statistics, cost analysis or numerical calculations. Some such techniques are listed below :

(1) Linear Programming	(2) Non-linear Programming	(3) Integer Programming
(4) Dynamic Programming	(5) Goal Programming	(6) Games Theory
(7) Inventory Control	(8) PERT-CPM	(9) Simulation.

(10) Queueing Theory etc. Here, for example, we describe in brief the *queueing* or *waiting line theory*.

Queues have become an integral part of our daily life. Queues are formed everywhere where a service is fered and the service rate is slower than the arrival rate of customers. People waiting for railway reservations, machines waiting for repairs at a workshop and aeroplanes waiting in the sky to find a place to land at the airport are all examples of queueing.

Costs are associated with the waiting in a line and costs are also associated with adding more service facilities or counters. The purpose of OR study is to decide the optimum number of service facilities so as to minimise the sum of waiting period cost and cost of providing facilities.

Queueing theory works out the expected number of people in the queue, expected waiting time in the queue, expected idle time for the server etc. These calculations then help in deciding the optimum number of service facilities under given constraints.

Q. 1. Write a note on application of various quantitative techniques in different fields of business decision making.

2. Explain various types of O.R. models and indicate their application to production, inventory, and distribution systems.

3. Enumerate six techniques of operations Research and describe one briefly. **[IGNOU 2001 (Jan)]**

3.14 DEVELOPMENT OF OPERATIONS RESEARCH IN INDIA

In 1949, Operations Research came into picture when an OR unit was established at the Regional Research Laboratory, Hyderabad. At the same time, Prof. R.S. Verma (Delhi University) setup an OR team in the Defence Science Laboratory to solve the problems of store, purchase and planning. In 1953, Prof. P.C. Mahalanobis established an OR team in the Indian Statistical Institute, Calcutta, for solving the problem of national planning and survey. In 1957, Operations Research Society of India was formed and this society became a member of the International Federation of Operations Research Societies in 1960. Presently India is publishing a number of research journals, namely, 'OPSEARCH', 'Industrical Engineering and Management', 'Materials Management Journal of India', 'Defence Science Journal', 'SCIMA', 'Journal of Engineering Production', etc.

As far as the OR education in India is concerned University of Delhi was the first to introduce a complete M.Sc. course in OR in 1963. Simultaneously, Institute of Management at Calcutta and Ahemdabad started teaching OR in their MBA courses. Now-a-days, OR has become so popular subject that it has been introduced in almost all Institutes and Universities in various disciplines like, Mathematics, Statistics, Commerce, Economics, Management Science, Medical science, Engineering, etc. Also, realizing the importance of OR in Accounts and Administration, government has introduced this subject for the IAS, CA, ICWA examinations, etc.

Prof. Mahalanobis first applied OR in India by formulating second five-year plan with the help of OR techniques. Planning Commission made the use of OR techniques for planning the optimum size of the Caravelle fleet of Indian air lines. Some of the industries, namely, Hindustan Lever Ltd.; Union Carbide, TELCO, Hindustan Steel, Imperial Chemical Industries, Tata Iron & Steel Company, Sarabhai Group, FCI, etc. have engaged OR teams. ***Kirlosker Company*** is using the assignment technique of OR to maximize profit.

Textile firms like, DCM., Binni's and Calico, etc., are using linear programming techniques. Among other Indian organizations using OR are the Indian Railways, CSIR, Tata Institute of Fundamental Research, Indian Institute of Science, State Trading Corporation, etc.

****It is also worthnoting that the present text on*** 'OPERATIONS RESEARCH' ***is the first book published in India to meet the requirements of various courses on this subject.***

3.15 ROLE OF COMPUTERS IN OPERATIONS RESEARCH

In fact, computers have played a vital role in the development of OR. But OR would not have achieved its present position for the use of computers. The reason is that—in most of the OR techniques computations are so complex and involved that these techniques would be of no practical use without computers. Many large scale applications of OR techniques which require only few minutes on the computer may take weeks, months and sometimes years even to yield the same results manually. So the computer has become as essential and integral part of OR. Now-a-days, OR methodology and computer methodology are growing up simultaneously. It seems that in the near future the line dividing the two methodologies will disappear and the two sciences will combine to form a more general and comprehensive science. It should also be noted that FORTRAN and C-programs are functionally equivalent.

The computor software packages are useful for rapid and effective calculations which is a necessary part of O.R. approach to solve the problems. These are :

(i) QSB+ (Quantitative System for Business Plus), Version 3.0, by Yih-long Chang and Robert S. Sullivan, is a software package that contains problem solving algorithms for OR/MS, as well as modules on basic statistics, non-linear programming and financial analysis.

(ii) QSOM (Quantitative Systems for Operations Management), by Yih-long, is an interactive user-friendly system. It contains problem-solving algorithms for operations management problems and associated information system.

(iii) Value STORM : MS quantitative Modelling for Decision Support, by Hamilton Emmons, A.D. Flowers, Chander Shekhar, M.Khot and Kamlesh Mathur, is a special version of Personal STORM version 3.0 developed for use in OR/MS.

(iv) Excel 97 by Gene Weiss Kopf and distributed by BPB publications, New Delhi, is an easy-to-use task-oriented guide to Excel Spread sheet applications.

(v) LINDO (Linear Interactive Discrete Optimization), developed by Linus Schrage Lindo in his book "An Optimization Modeling System, 4th ed. (Palo Alto, CA : Scientific Press 1991)

SELF-EXAMINATION QUESTIONS

1. (a) What is Operations Research ? A certain wine importer noticed that his sales of wine were not what they should be in comparison to other types of liquor. He hired you as a consultant to look into this problem, with the intention of improving the wine business. What would you do ?
   (b) How does one go about organising for effective Operations Research ? Explain.
2. Give a brief account of the methods used in model formulation.
3. Explain, how and why OR methods have been valuable in aiding executive decision. **[Meerut (Stat.) 90]**
4. Explain the concept, scope and tools of OR as applicable to business and industry.
5. Discuss the advantages and limitations of using results from a mathematical model to make decisions about operations.
6. "Mathematics of OR is mathematics of optimization". Discuss.
7. "OR is the application of scientific methods, techniques and tools to problem involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem." Discuss.
8. (a) Define Operations Research. Give the main characteristics of Operations Research.
   (b) Discuss the importance of Operations Research in decision-making process.
9. (a) Dicuss the significance and scope of Operations Research in modern management.
   (b) Describe in brief the uses of statistical techniques in Operations Research.
10. Write a detailed note on the use of models for decision-making. Your answers should specifically cover the following :
    (i) Need for model building.
    (ii) Type of model appropriate to the situation.
    (iii) Steps involved in the construction of the model.
    (iv) Setting up criteria for evaluating different alternatives.
    (v) Role of random numbers.
11. "Operations Research is an aid for the executive in making his decisions by providing him with the needed quantitative information, based on the scientific method analysis". Discuss this statement in detail, illustrating it with O.R methods that you know. **[Meerut (Stat.) 95]**
12. Give the essential characteristics of the following types of process :
    (a) Allocation (b) Competitive Games
    (c) Inventory (d) Waiting line.
13. What is the role of Operations Research in decision making ? Explain the scope and methodology of Operations Research, the main phases of Operations Research and techniques in solving an Operations Research problem.
14. Write short notes on the following :
    (i) Area of applications of Operations Research.
    (ii) Role of constraints and objectives in the construction of mathematical models.
    (iii) Statistician's role as member of O.R team.
15. Is Operations Research a discipline, or a profession, or set of techniques, or a philosophy, or a new name for an old thing ?
16. Outline broad features of the judgment phase and the research phase of scientific method in Operations Research. Discuss fully any one of these phases.
17. How can Operations Research models be classified ? Which is the best classification in terms of learning and understanding the fundamentals of Operations Research ?
18. What are the advantages and disadvantages of Operational Research models ? Why is it necessary to test models and how would you go about testing a model ?
19. What is Operations Research ? Describe four models used in Operations Research.
20. List any three Operations Research techniques and state in what conditions they can be used ?
21. Explain the role of quantitative techniques in the field of business and industry in modern times. Give a few examples in support of your answer.
22. What are the essential characteristics of Operations Research ? Mention different phases in an Operations Research study. Point out some limitations of Operations Research.
23. (a) Define OR as a decision making science.
    (b) Briefly explain the uses of OR-techniques in India. How are they found useful by the business executives ? Which of the three techniques are most commonly used in India ? Why
24. Explain the meaning and nature of OR.

25. State any four areas for the application of OR techniques in Financial Management, and how it improves the performance of the organisation.
26. (a) Comment on "Operations Research is a scientific and for enhancing creative and judicious capabilities of a decision maker".
    (b) Give any four processes of Operations Research and discuss their essential features.
27. Write a critical essay on the definition and scope of Operations Research.
28. Comment on the following statements :
    (a) OR is a bunch of mathematical techniques.
    (b) OR is no more than a quantitative analysis of the problem.
    (c) OR advocates a system's approach and is concerned with optimization. It provides a quantitative analysis for decision making.
    (d) OR has been defined semi-facetiously as the application of big minds to small problems.
29. What is Operations Research ? What areas of Operations Research have made a significant impact on decision making process ? Why is it important to keep an open mind in utilizing Operations Research techniques ?
30. Give a definition of Operations Research indicating the different types of models of the problem and the general methods of their solution.
31. (a) Write briefly about the following :
    (i) Iconic models, (ii) Analogue models, (iii) Mathematical models (or Symbolic models).
    (b) Explain three types of models used in Operations Research, giving suitable example.
    (c) What is the function of a model in decision making ? Name the types of models. What are the advantages of models? What are the pitfalls of models.
32. Distinguish the following models with suitable examples :
    (i) Stochastic and deterministic models; (ii) Static and dynamic models.
33. Quantitative techniques complement the experience and judgement of an executive in decision making. They do not and cannot replace it. Discuss.
34. In construction any OR model, it is essential to realize that a most important purpose of the modelling process is "to help any manager better." Keeping this purpose in mind, state any four OR models that can be of help to Chartered Accountants in advising their clients. **[C.A. (May) 91]**
35. State three properties and three advantages of an OR model. **[C.A. (May) 92]**
36. Describe briefly the components of a proplem and mention the three major types of problems in decision making under different environment. **[C.A. (Nov.) 92]**
37. "Much of the success of OR applications in the last three decades is due to the computers." Discuss. **[C.A. (May) 93]**
38. Discuss the role and scope of quantitative methods for scientific decision making in a business environment. **[IPM (MBA) 2000]**
39. Explain briefly the various applications of O.R. **[VTU (BE Compu.) Aug. 2001]**
40. What are the advantages and limitations of OR studies ? **[VTU (BE VIth Sem.) Feb. 2002]**

MODEL OBJECTIVE QUESTIONS

1. Operations research approach is
   (a) multi-disciplinary. (b) scientific. (c) intuitive. (d) all of the above.
2. Operations research analysts do not
   (a) predict future operations. (b) build more than one model.
   (c) collect relevant data. (d) recommend decision and accept.
3. For analysing a problem, decision-makers should normally study
   (a) its qualitative aspects. (b) its quantitative aspects.
   (c) both (a) and (b). (d) neither (a) nor (b).
4. Decision variables are
   (a) controllable. (b) uncontrollable. (c) parameters. (d) none of the above.
5. A model is
   (a) an essence of reality. (b) an approximation. (c) an idealization. (d) all of the above.
6. Managerial decisions are based on
   (a) an evaluation of quantitative data. (b) the use of qualitative factors.
   (c) numbers produced by formal models. (d) all of the above.
7. The use of decision models
   (a) is possible when the variable's value is known.
   (b) reduces the scope of judgement and intuition known with certainty in decision-making.

(c) requires the knowledge of computer software use.
(d) none of the above.

8. Every mathematical model
(a) must be deterministic.
(b) requires computer aid for its solution.
(c) represents data in numerical form.
(d) all of the above.

9. A physical model is example of
(a) an iconic model.
(b) an analogue model.
(c) a verbal model.
(d) a mathematical model.

10. An optimization model
(a) mathematically provides the best decision.
(b) provides decision within its limited context.
(c) helps in evaluating various alternatives constantly.
(d) all of the above.

11. Consider the following phases
1. Informationi phase
2. Evaluation phase
3. Creative phas॰
4. Investigation phase

The correct sequence of thse in value engineering is
(a) 1, 3, 4, 2
(b) 3, 1, 4, 2
(c) 3, 1, 4, 2
(d) 3, 1, 2, 4

[IES (Mech.) 1994]

Answers

1. (a)	**2.** (a)	**3.** (c)	**4.** (a)	**5.** (d)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (a)
10. (a)	**11.** (b).							

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